

Pinchers exist

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Abstract

We construct surface diffusion flows that drive smooth closed embedded tori to pinchoff in finite time. The flow remains embedded on its maximal interval $[0, T)$ and develops an axial curvature singularity at a finite time T ; away from the axis it converges locally smoothly to a given terminal geometry. We further characterise the singularity profile: If $A(t)$ denotes the radius of the waist, then

$$A(t) = \{4\mu(T - t)\}^{1/4}(1 + o(1)), \quad A(t)^{-1}r(A(t)\zeta, t) \xrightarrow{C_{\text{io}}^\infty} U(\zeta).$$

Here U is a rigorous realisation of the classical fundamental positive even conical similarity profile first computed numerically by Wong, Miksis, Voorhees and Davis and subsequently analysed by Bernoff, Bertozzi and Witelski.

1 Introduction

The surface diffusion flow of a smooth closed immersed surface $\Sigma_t \subset \mathbb{R}^3$ is

$$\partial_t f = -(\Delta H)\nu, \tag{1.1}$$

where $H = (\kappa_1 + \kappa_2)/2$ is the mean curvature. The capillarity-driven law has its physical origins in Herring's theory of surface transport and Mullins's theory of thermal grooving [21, 36]; its geometric formulation was developed further by Cahn and Taylor [8]. Nichols and Mullins derived the evolution equation for surfaces of revolution [37]. Coleman, Falk and Moakher later observed numerical breakup of cylindrical bodies under that equation [10, 11]. Wong, Miksis, Voorhees and Davis first computed the universal conical similarity profile and its cone angle [46]. Bernoff, Bertozzi and Witelski developed the similarity theory, computed the fundamental and higher profiles, derived their far-field WKB structure and performed a formal and numerical stability analysis [5]. Mayer subsequently reported numerical curvature blow-up for compact axisymmetric surfaces, including toroidal examples [32].

The purpose of this paper is to complete a rigorous dynamical realisation of pinchoff asymptotic to the fundamental Wong–Bernoff–Bertozzi–Witelski numerical profile. New contributions to the analysis of this profile include a proof of its existence, positivity, exact tail and flux identities, and the estimates needed to compactify it.

Rigorous examples of a different phenomenon were obtained by Giga and Ito for planar curve diffusion and by Mayer and Simonett for hypersurface diffusion: an initially embedded solution can remain smooth while reaching a self-intersection and thereby lose embeddedness [17, 33]. See also Blatt [7]. To the best of our knowledge, the theorem below is the first rigorous construction of a finite-time curvature singularity for surface diffusion flow, and the first rigorous compact realisation of the classical axisymmetric conical profile.

Rigorous constructions of non-stationary self-similar solutions for surface diffusion remain comparatively scarce. The rough-data theory of Koch and Lamm [24] yields global analytic forward self-similar solutions for entire graphs emanating from one-homogeneous Lipschitz data of sufficiently small slope, and Du and Yip [14] proved their stability. Rybka and the author [40]

obtained complementary classification and rigidity results for solitons that are entire graphs over $\mathbb{R}$; for forward self-similar profiles, their linearity conclusions require additional hypotheses. Giga and Katayama [18] subsequently clarified that these results are compatible with the genuinely nonlinear small-slope forward self-similar profiles furnished by the Koch–Lamm theory. Related forward self-similar solutions for thermal-grooving boundary problems were constructed by Asai and Giga [2] and, with the nonlinear no-flux condition, by Asai and Kohsaka [3]; see also the multidimensional half-space theory of Giga and Katayama [19]. These works concern forward expanding solutions, $\Gamma_t = t^{1/4}\Gamma_*$, which smooth homogeneous graphical data. By contrast, the profile constructed here is a backward self-similar shrinker, $\Gamma_t = (T - t)^{1/4}\Gamma_*$, with the geometry of a positive two-ended axisymmetric neck. Accordingly, the similarity term has the opposite sign and, relative to the planar graphical problem, the equation for $r = U(z)$ contains the additional azimuthal principal-curvature contribution. Axisymmetric conical shrinkers of this type were previously identified through asymptotic analysis and numerical shooting by Bernoff, Bertozzi and Witelski [5]. To the best of our knowledge, Theorem 3.9 gives the first rigorous construction of such a positive conical shrinking profile and, more broadly, Theorem 1.2 furnishes the first rigorous example of finite-time singularity formation for the surface diffusion flow, realised by a compact embedded trajectory and modelled by this profile.

The lack of a comparison principle makes this construction delicate. A noncompact similarity profile is not itself a solution arising from a closed surface: its conical ends must be joined to a compact outer cap, the volume constraint must be respected, and every non-decaying inner mode must be selected carefully, as there do exist unstable modes.

We first make precise the freedom in the compact outer geometry.

Definition 1.1 (Admissible outer cap). Fix $\alpha, \delta > 0$. An (α, δ) -admissible terminal meridian is a reflection-symmetric simple closed curve Γ_0 in the (z, r) half-plane $\{r \geq 0\}$ with the following properties:

- (i) its only point on the rotation axis is $p_0 = (0, 0)$;
- (ii) in the fixed conical collar $0 < |z| < 4\delta$ it is the graph $r_0(z) = \alpha|z|$;
- (iii) it is smooth and embedded away from p_0 , and on the complement of $|z| < 2\delta$ it has positive distance from the axis and a positive normal injectivity radius.

For a single smooth embedded outer arc, the two positivity requirements in (iii) follow from compactness. They are recorded explicitly because they must be uniform when the cap is varied. An admissible cap perturbation is a sufficiently small reflection-symmetric C^{k+8} normal graph, $k \geq 10$, supported outside $|z| < 4\delta$.

Theorem 1.2 (Finite-time conical pinch). *There are numbers $\mu, \alpha > 0$ and a smooth positive even conical profile U with the following property. Let Γ_0 be any (α, δ) -admissible terminal meridian. There is $A_* = A_*(\Gamma_0, \delta) > 0$ such that, for every $0 < A_0 < A_*$, the fixed-volume compatible trace class at waist radius A_0 contains an infinite-dimensional C^1 family $\mathcal{M}_{\text{pin}}$ of initial tori. Every smooth datum in $\mathcal{M}_{\text{pin}}$ generates a smooth surface diffusion flow*

$$f : \mathbb{T}^2 \times [0, T) \longrightarrow \mathbb{R}^3$$

which is rotationally symmetric, reflection-symmetric and embedded for $t < T$, and for which

$$\limsup_{t \uparrow T} \|\mathbf{A}(\cdot, t)\|_{L^\infty} = \infty,$$

where $\mathbf{A}$ is the second fundamental form. Near the reflected waist the surface is represented by a positive even axisymmetric graph $r = r(z, t)$. With $A(t) = r(0, t)$, every member of the family

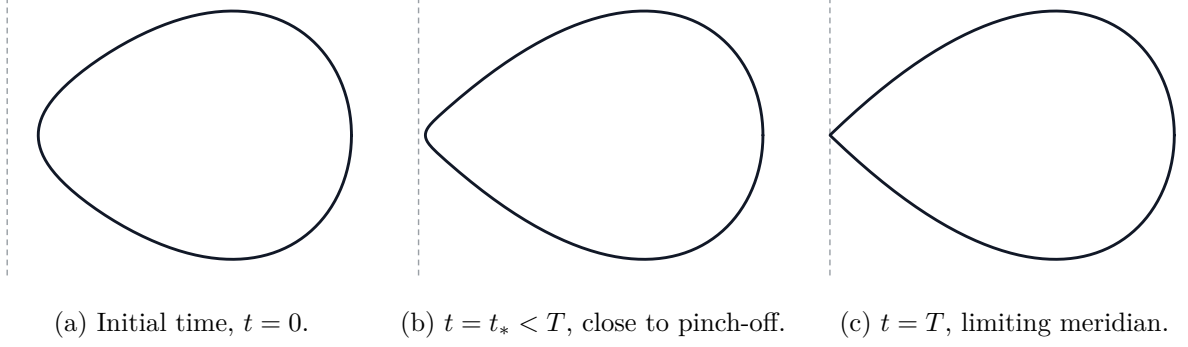


Figure 1: Illustrative meridian profiles for the conical-pinch family. The outer arc is schematic: the theorem permits any admissible cap in the sense of [Definition 1.1](#).

satisfies

$$A(t) = \{4\mu(T - t)\}^{1/4}(1 + o(1)), \quad (1.2)$$

$$\frac{r(A(t)\zeta, t)}{A(t)} \longrightarrow U(\zeta) \quad \text{in } C_{\text{loc}}^\infty(\mathbb{R}). \quad (1.3)$$

The profile satisfies

$$\mathcal{F}[U] + \mu(U - \zeta U') = 0, \quad U(0) = 1, \quad (1.4)$$

where $\mathcal{F}$ is the rotational surface-diffusion operator defined in [\(2.12\)](#), and has the conical expansion

$$U(\zeta) = \alpha|\zeta| - \frac{|\zeta|^{-3}}{8\mu\alpha(1 + \alpha^2)} - \frac{21(11\alpha^2 + 1)}{128\mu^2\alpha^3(1 + \alpha^2)^3}|\zeta|^{-7} + O(|\zeta|^{-11}). \quad (1.5)$$

Moreover,

$$\begin{aligned} |\mu - 0.030292801498271463| &\leq 1.361237725845061 \times 10^{-15}, \\ |\alpha - 1.037079401503446| &\leq 1.557172962297049 \times 10^{-12}. \end{aligned}$$

The flow converges smoothly on compact subsets of the limiting meridian separated from the terminal conical point. Its limiting outer cap is a small smooth perturbation of the prescribed reference cap Γ_0 .

Let E_c be the finite-dimensional controlled spectral space in [Proposition 4.2](#) and put $N = \dim E_c$. At every fixed Sobolev level $k \geq 10$, $\mathcal{M}_{\text{pin}}$ has codimension N inside the fixed-volume compatible trace class, and codimension $N + 1$ in the unrestricted compatible trace class. The family and the limiting outer cap depend C^1 on admissible perturbations of Γ_0 . The smallness threshold A_* is chosen, in particular, so that the physical time $A_0^4/(2\mu)$, which bounds the remaining time after shrinking the construction, lies inside the uniform outer-cap existence interval furnished by [Lemma 5.2](#).

The geometry of the degeneration is illustrated in [Figure 1](#).

There are two independent difficulties. First, the similarity equation is a singular fourth-order boundary-value problem: regularity and evenness at the waist leave two shooting parameters, while a conical end must contain neither of the two growing WKB components. We solve this by a finite-base Volterra chart at infinity and a validated Poincaré–Miranda match to the regular centre. The resulting profile obeys the exact relative-volume cancellation

$$\int_0^\infty (U^2 - \alpha^2 \zeta^2) \, d\zeta = 0.$$

This identity removes the potentially leading order- A^3 volume defect.

Second, the conical end produces outward transport in similarity variables. Consequently global exponential decay relative to the cone is false. We work instead in polynomially discounted weighted Sobolev spaces. The essential spectral boundary then lies strictly in the left half-plane, so only finitely many slow or growing core modes remain. A future Duhamel condition selects those modes, while compactly supported initial variations realise the required finite-dimensional compatibility conditions.

The stable datum is not consumed by this selection. The controlled component is a C^1 function of an arbitrary small datum in the infinite-dimensional free trace space. After one additional scalar is used to impose the conserved volume, the compact construction therefore gives a finite-codimensional (infinite dimensional) family. This is the stability proved here: persistence on a C^1 manifold of codimension N in the fixed-volume symmetric class. It is an open-set stability statement only in the special case $N = 0$.

The outer cap is matched at a fixed physical interface. Its first backward-time Taylor coefficient agrees exactly with the forced inverse-cubic tail of U . After straightening the dilation, a uniform convective maximal-regularity estimate gives the inner inverse; ordinary maximal regularity on a closed double gives the outer inverse. The local representatives are required to describe one global normal height on their overlap. A physical-collar estimate then joins the two local inverses into a uniform fibre-product parametrix, and a contraction argument produces an exact compact invariant curve.

The word “any” in the choice of outer cap is always understood in the admissible sense of [Definition 1.1](#). No global curvature smallness is imposed on one fixed cap. Instead, smoothness and embeddedness give a positive outer normal-graph radius and a positive short-time parabolic existence interval. The remaining physical time is $A_0^4/(4\mu)\{1 + o(1)\}$ and, after shrinking the construction, is bounded by $A_0^4/(2\mu)$. It can therefore be made shorter than that outer interval. The contraction then keeps the actual outer arc in the normal-graph neighbourhood throughout the evolution, so it cannot develop a separate singularity before the conical neck pinches. Uniform cap families require uniform distance-to-axis, reach and C^{k+8} bounds. The reference cap is not solved backwards from $t = T$; the construction produces a nearby terminal cap as part of the global forward solution.

The theorem concerns a finite-codimensional family of specially selected initial data. It does not assert that a circular torus enters this family, nor that the profile is spectrally stable. Those are strictly stronger capture questions.

The paper is organised as follows. [Section 2](#) records the axisymmetric equation and terminal scaling. The positive profile and its two isolated computer-assisted validations are treated in [Section 3](#). Weighted terminal dynamics and finite-mode selection are treated in [Section 4](#). The fixed-interface compactification is proved in [Section 5](#). The detailed end parametrix appears in [Section A](#).

2 Geometric setting and terminal scaling

Let $f : M^2 \rightarrow \mathbb{R}^3$ be a smooth immersion of a closed orientable surface. We write g for the induced metric, ν for a local unit normal, $\mathbf{A}$ for the second fundamental form, and

$$H = \frac{1}{2} \operatorname{tr}_g \mathbf{A}$$

for the averaged scalar mean curvature. Our Laplace–Beltrami operator is non-positive in L^2 :

$$\int_M \varphi \Delta \varphi \, d\mu = - \int_M |\nabla \varphi|^2 \, d\mu. \quad (2.1)$$

The surface diffusion flow is

$$\partial_t f = -(\Delta H)\nu. \quad (2.2)$$

Changing the local orientation changes both H and ν and leaves the vector field in (2.2) unchanged.

Lemma 2.1 (Area and volume). *Let f_t be a smooth solution of (2.2). Then*

$$\frac{d}{dt} \text{Area}(\Sigma_t) = -2 \int_{\Sigma_t} |\nabla H|^2 d\mu. \quad (2.3)$$

If Σ_t is embedded and bounds a region Ω_t , then

$$\frac{d}{dt} \text{Vol}(\Omega_t) = 0. \quad (2.4)$$

Proof. For a normal velocity $V = \langle \partial_t f, \nu \rangle$, the first variations in the present convention are

$$\frac{d}{dt} \text{Area}(\Sigma_t) = -2 \int_{\Sigma_t} HV d\mu, \quad \frac{d}{dt} \text{Vol}(\Omega_t) = \int_{\Sigma_t} V d\mu.$$

Substituting $V = -\Delta H$, integrating by parts, and using that Σ_t is closed gives both identities. $\square$

We use the local well-posedness theory in normal-graph coordinates. Let Σ be a smooth compact orientable immersed reference surface with immersion $i : \Sigma \rightarrow \mathbb{R}^3$ and smooth unit normal ν_Σ . For a sufficiently small function ρ define

$$i_\rho(p) = i(p) + \rho(p)\nu_\Sigma(p), \quad \Gamma_\rho = i_\rho(\Sigma). \quad (2.5)$$

We write $h^s(\Sigma)$ for the little Hölder space of order s .

Theorem 2.2 (Local normal-graph semiflow). *Let $0 < \alpha < 1$ and let Σ be as above. Given a sufficiently small $\sigma \in h^{2+\alpha}(\Sigma)$, there exist $T_0 > 0$ and an $h^{2+\alpha}$ neighbourhood $\mathcal{W}$ of σ such that, for every $\rho_0 \in \mathcal{W}$, surface diffusion has a unique classical solution represented by a normal graph*

$$\rho(\cdot; \rho_0) \in C([0, T_0], h^{2+\alpha}(\Sigma)) \cap C^\infty((0, T_0) \times \Sigma).$$

The map $(t, \rho_0) \mapsto \rho(t; \rho_0)$ is a smooth local semiflow. If ρ_0 is smooth, then the solution is differentiable at $t = 0$ as a C^0 -valued map and

$$\partial_t \rho(0; \rho_0) = \mathcal{G}(\rho_0), \quad (2.6)$$

where $\mathcal{G}$ is the surface diffusion graph operator. More precisely, the solution has the standard $C([0, T_0], h^{4+\alpha_0}) \cap C^1([0, T_0], h^{\alpha_0})$ regularity for every exponent $\alpha_0 < \alpha$ for which $\rho_0 \in h^{4+\alpha_0}$.

Proof. This is the normal-graph formulation of the local existence, uniqueness and smooth-dependence theorem of Escher, Mayer and Simonett [15]; the precise semiflow statement used here is also recorded in [33, Proposition 2.1]. We use it as the standard local theory for the quasilinear fourth-order equation. For the corresponding axisymmetric well-posedness and bifurcation framework, see also [26]. $\square$

Lemma 2.3 (Preservation of symmetries). *Suppose the initial immersion and reference parametrisation are invariant under a compact group of Euclidean isometries and domain diffeomorphisms. Then the unique local solution from Theorem 2.2 has the same symmetry for as long as the normal-graph representation exists.*

Proof. Apply the symmetry to the solution. Euclidean invariance of (2.2) gives another solution with the same initial data. Uniqueness gives equality. $\square$

2.1 Axisymmetric graphs

Consider an axisymmetric graph

$$f(z, \theta, t) = (r(z, t) \cos \theta, r(z, t) \sin \theta, z), \quad r > 0. \quad (2.7)$$

Set

$$E = 1 + r_z^2$$

and choose

$$\nu = \frac{1}{\sqrt{E}}(-\cos \theta, -\sin \theta, r_z). \quad (2.8)$$

The induced metric is $E \, dz^2 + r^2 \, d\theta^2$. A direct calculation gives

$$H = \frac{1}{2} \left(-\frac{r_{zz}}{E^{3/2}} + \frac{1}{r\sqrt{E}} \right). \quad (2.9)$$

For an axisymmetric scalar function $\varphi(z)$,

$$\Delta \varphi = \frac{1}{r\sqrt{E}} \partial_z \left(\frac{r}{\sqrt{E}} \varphi_z \right). \quad (2.10)$$

Since $\langle f_t, \nu \rangle = -r_t/\sqrt{E}$, equation (2.2) is equivalent to

$$r_t = \frac{1}{r} \partial_z \left(\frac{r}{\sqrt{E}} \partial_z H \right), \quad (2.11)$$

with H given by (2.9).

This is the classical surface-of-revolution equation of Nichols and Mullins [37], written in the present averaged-mean-curvature and sign conventions. It is also the equation used in the early numerical and similarity studies [10, 11, 46, 5].

Remark 2.4. Rotational symmetry and reflection symmetry $z \mapsto -z$ are preserved by Lemma 2.3. They do not, by themselves, imply that the evolving surface remains a graph of the form (2.7), nor that $z = 0$ remains the global minimum of cylindrical radius. Every neck statement below is therefore understood up to the first time at which the indicated graph or neck conditions fail. For the orbit constructed in this paper, preservation of the graph, positivity and embeddedness is verified quantitatively in Lemma 5.10.

2.2 Terminal similarity variables

For a positive profile Q define

$$\mathcal{F}[Q] := \frac{1}{2Q} \partial_\zeta \left[\frac{Q}{\sqrt{1 + Q_\zeta^2}} \partial_\zeta \left(\frac{1}{Q\sqrt{1 + Q_\zeta^2}} - \frac{Q_{\zeta\zeta}}{(1 + Q_\zeta^2)^{3/2}} \right) \right]. \quad (2.12)$$

The scaling identity associated with (2.11) is

$$\mathcal{F}[AQ(\cdot/A)](z) = A^{-3} \mathcal{F}[Q](z/A).$$

As long as the reflected waist radius $A(t) = r(0, t)$ is positive, put

$$\zeta = \frac{z}{A(t)}, \quad P(\zeta, s) = \frac{r(z, t)}{A(t)}, \quad \frac{ds}{dt} = A(t)^{-4}. \quad (2.13)$$

Then $P(0, s) = 1$ and

$$P_s = \mathcal{F}[P] - \mathcal{F}[P](0)(P - \zeta P_\zeta), \quad (2.14)$$

$$(\log A)_s = \mathcal{F}[P](0). \quad (2.15)$$

Indeed, at fixed z ,

$$r_t = A_t(P - \zeta P_\zeta) + A^{-3}P_s,$$

and evaluation at $\zeta = 0$ gives $A_t = A^{-3}\mathcal{F}[P](0)$.

A stationary normalised profile with $\mathcal{F}[U](0) = -\mu < 0$ therefore satisfies

$$\mathcal{F}[U] + \mu(U - \zeta U') = 0, \quad U(0) = 1, \quad \mu > 0. \quad (2.16)$$

The associated noncompact similarity solution has

$$A_t = -\mu A^{-3}, \quad A(t) = \{4\mu(T - t)\}^{1/4}. \quad (2.17)$$

To compare normalisations, let R_0 denote the fundamental profile in [5]. Those authors use $H_{\text{BBW}} = \kappa_1 + \kappa_2 = 2H$, so, after matching normal orientations, their axisymmetric operator is $2\mathcal{F}$ and their profile equation is $\mathcal{F}[R] + \frac{1}{8}(R - \eta R') = 0$. Consequently the profiles are related by

$$R_0(\eta) = cU(\eta/c), \quad c = (8\mu)^{1/4}. \quad (2.18)$$

The enclosure proved below gives $c = 0.701628717842213\dots$ and $\alpha = 1.037079401503446\dots$, hence the half-angle $\arctan \alpha = 46.042796186\dots^\circ$. These values are consistent with Table I of [5], which reports $R_0(0) = 0.701595$, slope 1.03714 and half-angle 46.0444° . The remainder of the paper constructs a compact trajectory asymptotic to this model.

3 Rigorous construction of the classical positive conical profile

The similarity equation and its fundamental conical solution were obtained numerically in [46, 5]. The results below prove existence and positivity with outward-rounded bounds and derive the exact tail, flux, volume and dissipation identities needed for compactification. The conical far field imposes more than the leading slope. Under the differentiated polyhomogeneous ansatz below, the first nontrivial algebraic correction is forced by the similarity equation.

Proposition 3.1 (Forced conical tail). *Let $\mu > 0$, let P solve (2.16) on a positive half-line, and suppose that, as $\zeta \rightarrow \infty$,*

$$P(\zeta) = \alpha\zeta + b\zeta^{-3} + O(\zeta^{-7}), \quad \alpha > 0, \quad (3.1)$$

with the corresponding differentiated expansions through order four. Then

$$b = -\frac{1}{8\mu\alpha(1 + \alpha^2)} < 0. \quad (3.2)$$

If in addition

$$P(\zeta) = \alpha\zeta + b\zeta^{-3} + c\zeta^{-7} + O(\zeta^{-11})$$

with differentiated asymptotics, then

$$c = -\frac{21(11\alpha^2 + 1)}{128\mu^2\alpha^3(1 + \alpha^2)^3}. \quad (3.3)$$

If P extends smoothly and evenly across $\zeta = 0$, (2.16) also has the integrated form

$$\begin{aligned} \frac{P}{\sqrt{1 + P_\zeta^2}} \partial_\zeta \left(\frac{1}{P\sqrt{1 + P_\zeta^2}} - \frac{P_\zeta \zeta}{(1 + P_\zeta^2)^{3/2}} \right) \\ = \mu \zeta P^2 - 3\mu \int_0^\zeta P(\eta)^2 \, d\eta. \end{aligned} \quad (3.4)$$

Proof. For the ansatz in (3.1) one has

$$\mathcal{F}[P] = \frac{1}{2\alpha(1+\alpha^2)}\zeta^{-3} + O(\zeta^{-7}).$$

In particular, the coefficient b contributes only at order ζ^{-7} to this expression, whereas

$$P - \zeta P_\zeta = 4b\zeta^{-3} + O(\zeta^{-7}).$$

Equation (3.2) follows from (2.16). Expanding the same equation through order ζ^{-7} gives (3.3). To obtain (3.4), multiply (2.16) by $2P$ and integrate from zero to ζ . Evenness makes the flux vanish at zero, and

$$2P(P - \zeta P_\zeta) = 3P^2 - (\zeta P^2)_\zeta.$$

□

Proposition 3.2 (Flux system and volume neutrality). *Let U be a smooth positive even solution of (2.16), normalised by $U(0) = 1$, and suppose that it has the differentiated conical expansion*

$$U(\zeta) = \alpha\zeta + b\zeta^{-3} + O(\zeta^{-7}) \quad (\zeta \rightarrow +\infty).$$

Put

$$q = U', \quad S = (1 + q^2)^{1/2}, \quad K = \frac{1}{US} - \frac{q'}{S^3}, \quad J = \frac{U}{S}K'.$$

Then the similarity equation is equivalent on $[0, \infty)$ to

$$U' = q, \quad q' = \frac{1 + q^2}{U} - (1 + q^2)^{3/2}K, \quad (3.5)$$

$$K' = \frac{\sqrt{1 + q^2}}{U}J, \quad J' = -2\mu U(U - \zeta q), \quad (3.6)$$

with

$$U(0) = 1, \quad q(0) = J(0) = 0. \quad (3.7)$$

Moreover,

$$J(\zeta) = \mu \left\{ \zeta U(\zeta)^2 - 3 \int_0^\zeta U(\eta)^2 \, d\eta \right\}, \quad (3.8)$$

and every such conical profile satisfies the exact cancellation

$$\boxed{\int_0^\infty (U(\zeta)^2 - \alpha^2 \zeta^2) \, d\zeta = 0.} \quad (3.9)$$

Proof. The definition of K gives the equation for q' , while the definition of J gives the equation for K' . Since

$$\mathcal{F}[U] = \frac{1}{2U}J',$$

the similarity equation gives the final equation in (3.6). Evenness gives (3.7). Integration from zero, followed by integration by parts in the term containing $\eta U U'$, yields (3.8).

Set $d(\zeta) = U(\zeta)^2 - \alpha^2 \zeta^2$. The conical expansion gives $d(\zeta) = 2ab\zeta^{-2} + O(\zeta^{-6})$, so d is integrable at infinity. The differentiated expansion also gives

$$J(\zeta) = -\frac{1}{1 + \alpha^2}\zeta^{-1} + O(\zeta^{-5}),$$

so in particular $J(\zeta) \rightarrow 0$. The pure conical terms cancel in (3.8), and hence

$$\frac{J(\zeta)}{\mu} = \zeta d(\zeta) - 3 \int_0^\zeta d(\eta) \, d\eta.$$

The left side tends to zero, as does $\zeta d(\zeta)$. Passing to the limit proves (3.9). □

Proposition 3.3 (Relative area and dissipation identity). *Let U satisfy the hypotheses of Proposition 3.2, put $H_U = K/2$, and define*

$$\mathcal{A}_{\text{ren}}[U] := 2\pi \int_{\mathbb{R}} \left\{ U(1 + U'^2)^{1/2} - \alpha(1 + \alpha^2)^{1/2} |\zeta| \right\} d\zeta, \quad (3.10)$$

$$\mathcal{D}[U] := 2\pi \int_{\mathbb{R}} \frac{U}{(1 + U'^2)^{1/2}} |H'_U|^2 d\zeta. \quad (3.11)$$

Then both integrals are finite and

$$\boxed{\mathcal{D}[U] = \mu \mathcal{A}_{\text{ren}}[U] > 0.} \quad (3.12)$$

Proof. The conical expansion gives convergence. For

$$U_\lambda(\zeta) = \lambda U(\zeta/\lambda)$$

the limiting cone is unchanged and

$$\mathcal{A}_{\text{ren}}[U_\lambda] = \lambda^2 \mathcal{A}_{\text{ren}}[U].$$

The variation at $\lambda = 1$ is $DU = U - \zeta U'$. Direct first variation and integration by parts give

$$\begin{aligned} 2\mathcal{A}_{\text{ren}}[U] &= 2\pi \int_{\mathbb{R}} \left\{ S DU + \frac{UU'}{S} (DU)' \right\} d\zeta \\ &= 2\pi \int_{\mathbb{R}} U K DU d\zeta = 4\pi \int_{\mathbb{R}} U H_U DU d\zeta. \end{aligned}$$

The boundary term vanishes by the differentiated conical expansion. Since

$$\mathcal{F}[U] = \frac{1}{U} \left(\frac{U}{S} H'_U \right)' = -\mu DU,$$

multiplication by UH_U and a second integration by parts yield

$$\int_{\mathbb{R}} \frac{U}{S} |H'_U|^2 d\zeta = \mu \int_{\mathbb{R}} U H_U DU d\zeta.$$

This proves (3.12). If the left side vanished, then H_U would be constant and the profile equation would give $DU \equiv 0$, contradicting $U(0) = 1$. Hence both sides are strictly positive. $\square$

Corollary 3.4 (Absence of an order- A^3 volume obstruction). *Replace the cone $r = \alpha|z|$ on $0 \leq z \leq \delta$ by $r = AU(z/A)$, where $\delta > 0$ is fixed and $A/\delta \rightarrow 0$. The one-sided $r^2 dz$ -moment discrepancy is*

$$A^3 \int_0^{\delta/A} (U^2 - \alpha^2 \zeta^2) d\zeta = -\frac{2\alpha b}{\delta} A^4 + O_U(A^8 \delta^{-5}). \quad (3.13)$$

For the even two-sided replacement, the algebraic three-volume discrepancy is the corresponding fixed signed multiple and is therefore $O(A^4)$. Any matching interpolation with the same fourth-order scale adds $O_{\delta,U}(A^4)$ and cannot create an order- A^3 term. Choose a smooth reflection-even outer normal variation $\phi\nu$ on the reference outer surface Σ_* , supported away from the core, with $\int_{\Sigma_*} \phi d\mu \neq 0$. The implicit function theorem supplies a unique variation amplitude $\theta(A) = O(A^4)$ which restores the prescribed volume. No order- A^3 correction is required.

Proof. Use (3.9) to replace the truncated integral by minus the tail integral and then use $U^2 - \alpha^2 \zeta^2 = 2\alpha b \zeta^{-2} + O(\zeta^{-6})$. $\square$

For the higher tail coefficients used below, define recursively

$$c_n(\alpha, \mu) = -\frac{1}{4n\mu}[\zeta^{1-4n}]\mathcal{F}\left[\alpha\zeta + \sum_{j=1}^{n-1} c_j(\alpha, \mu)\zeta^{1-4j}\right]. \quad (3.14)$$

Here $[\zeta^m]$ denotes extraction of the coefficient of ζ^m in the formal Laurent expansion at infinity. This agrees with (3.2) and (3.3) for $n = 1, 2$.

The leading conical mode and homogeneous WKB splitting—one decaying and two growing exponential branches—were derived formally in [5, Section 4.1]. The finite-base Volterra chart, differentiated remainder bounds and validated matching used here are the rigorous additions.

Proposition 3.5 (Finite-base conical tail chart). *Let $\mathcal{P} \subseteq (0, \infty)^2$ be a compact set of parameters (α, μ) , and let $N \geq 1$. There is $L_0 < \infty$ with the following property. For every $L \geq L_0$, every $(\alpha, \mu) \in \mathcal{P}$, and every sufficiently small real number C_L , there is a unique solution*

$$U_\infty(\cdot; \alpha, \mu, C_L)$$

of (2.16) on $[L, \infty)$ which has conical slope α , contains no growing WKB component, and has finite-base stable coordinate C_L in the normalisation below. It is positive, has the differentiated expansion

$$U_\infty(\zeta) = \alpha\zeta + \sum_{n=1}^N c_n(\alpha, \mu)\zeta^{1-4n} + O(\zeta^{-4N-3}),$$

and bootstraps to the full polyhomogeneous expansion. The map

$$(\alpha, \mu, C_L) \mapsto (U_\infty, q_\infty, K_\infty, J_\infty)(L)$$

is C^1 .

Proof. Put $s_\alpha^2 = 1 + \alpha^2$. The exact linearisation of (2.16) at $U = \alpha\zeta$ is

$$\mathcal{A}_\alpha v = -\frac{v^{(4)}}{2s_\alpha^4} - \frac{v'''}{s_\alpha^4\zeta} + \frac{\alpha^2 - 1}{2\alpha^2 s_\alpha^4 \zeta^2} v'' + \frac{v'}{\alpha^2 s_\alpha^4 \zeta^3} - \frac{3v}{2\alpha^2 s_\alpha^2 \zeta^4}. \quad (3.15)$$

Set

$$\kappa = \frac{3}{4}\{2\mu(1 + \alpha^2)^2\}^{1/3}, \quad t = \kappa\zeta^{4/3}, \quad v = \zeta^{-5/3}y(t).$$

Then $\mathcal{A}_\alpha v + \mu(v - \zeta v') = 0$ is equivalent to

$$y^{(4)} - \frac{2}{t}y''' + \frac{A_2}{t^2}y'' + \left(1 - \frac{A_1}{t^3}\right)y' + \left(-\frac{2}{t} + \frac{A_0}{t^4}\right)y = 0, \quad (3.16)$$

where

$$A_2 = \frac{47}{8} + \frac{9}{16\alpha^2}, \quad A_1 = \frac{771}{64} + \frac{135}{64\alpha^2}, \quad A_0 = \frac{3403}{256} + \frac{873}{256\alpha^2}.$$

The limiting characteristic roots are $0, -1, \rho_+, \rho_-$, where $\rho_\pm = (1 \pm i\sqrt{3})/2$.

Let

$$P_N(\zeta) = \alpha\zeta + \sum_{n=1}^N c_n(\alpha, \mu)\zeta^{1-4n}, \quad y(t) = \zeta^{5/3}\{U(\zeta) - P_N(\zeta)\},$$

put $X = (y, y_t, y_{tt}, y_{ttt})^T$, and let V have columns $(1, \lambda, \lambda^2, \lambda^3)^T$ in the order $0, -1, \rho_+, \rho_-$. For $z = V^{-1}X = (z_0, z_s, z_+, z_-)$, direct substitution gives

$$z' = \text{diag}(0, -1, \rho_+, \rho_-)z + \frac{1}{t}B_1z + \mathcal{R}_N(t, z), \quad (3.17)$$

with

$$B_1 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ -2/3 & 0 & 0 & 0 \\ -2/3 & 0 & 0 & 0 \\ -2/3 & 0 & 0 & 0 \end{pmatrix}, \quad |\mathcal{R}_N(t, z)| \leq C_{\mathcal{P}} \{t^{-2}|z| + t^{-1}|z|^2 + t^{-3N-2}\}. \quad (3.18)$$

The last term follows because the coefficients $c_1, \dots, c_N$ cancel the algebraic residual to the corresponding order.

Let $t_0 = \kappa L^{4/3}$. Move $2z_0/t$ to the centre diagonal and call the remaining right side $\mathcal{Q}$. Fixing the slope, killing the growing WKB modes and imposing $z_s(t_0) = C_L$ are equivalent to

$$z_0(t) = -t^2 \int_t^\infty s^{-2} \mathcal{Q}_0(s, z(s)) \, ds, \quad (3.19)$$

$$z_s(t) = e^{-(t-t_0)} C_L + \int_{t_0}^t e^{-(t-s)} \mathcal{Q}_s(s, z(s)) \, ds, \quad (3.20)$$

$$z_{\pm}(t) = - \int_t^\infty e^{\rho_{\pm}(t-s)} \mathcal{Q}_{\pm}(s, z(s)) \, ds. \quad (3.21)$$

Thus C_L is a finite-base coordinate, not an asymptotic Stokes normalisation.

Before applying the fixed-point theorem, increase L_0 and restrict the chart ball so that its entire tube satisfies

$$U(\zeta) \geq \frac{1}{2} \alpha_{\min} \zeta, \quad \alpha_{\min} = \min_{(\alpha, \mu) \in \mathcal{P}} \alpha.$$

The rational profile vector field is therefore regular throughout the tube. Put $r = 3N$, separate the free term $C_L e^{-(t-t_0)}$, and use

$$\|z\|_{r, t_0} = \sup_{t \geq t_0} \left(\frac{t}{t_0} \right)^r \max \left\{ \frac{t}{t_0} |z_0(t)|, |z_s(t)|, |z_+(t)|, |z_-(t)| \right\}.$$

Work on the real subspace $z_- = \bar{z}_+$. For $t_0 > r$ the three Volterra kernels are bounded on this anisotropic space. The t^{-2} linear remainder maps a WKB component into the centre component with norm $O(t_0^{-1})$; the triangular B_1/t term maps the centre component back into the WKB block with the same gain. The corrected quadratic remainder is $O(t^{-1}|z|^2)$; its Lipschitz constant is controlled by the chart-ball radius, which is made small together with C_L and the algebraic forcing. The linear block feedback tends to zero as $t_0 \rightarrow \infty$. The contraction mapping theorem gives existence and uniqueness. For parameter dependence first translate the moving half-line by $\tau = t - t_0(\alpha, \mu)$; the uniform parameter-dependent contraction theorem then gives C^1 dependence.

The Volterra equations bootstrap the fixed-point bound to $z_0 = O(t^{-r-1})$ and the non-free WKB components $O(t^{-r-2})$. Since $r = 3N$, returning to U proves the displayed $O(\zeta^{-4N-3})$ expansion; iteration of the equation gives all differentiated orders. The same tube bound records positivity of every chart member. $\square$

Proposition 3.6 (Finite centre–tail matching certificate). *Let $Y = (U, q, K, J)$ denote the flux variables. For centre parameters (γ, μ) , let $Y_c(\cdot; \gamma, \mu)$ solve (3.5)–(3.6) with*

$$Y_c(0; \gamma, \mu) = (1, 0, 1 - \gamma, 0). \quad (3.22)$$

Take $L \geq L_0$ in Proposition 3.5, and let $Y_\infty(\cdot; \alpha, \mu, C_L)$ be that chart. Define

$$G(\gamma, \mu, \alpha, C_L) = Y_c(L; \gamma, \mu) - Y_\infty(L; \alpha, \mu, C_L). \quad (3.23)$$

If a validated degree argument—for example interval Newton [35], the Krawczyk operator [25], or the Poincaré–Miranda theorem [34]—proves that G has a zero in a box on which the centre and tail solutions remain positive, then (2.16) has a smooth positive even conical solution.

Proof. At a zero of G , the centre and tail solutions agree in all four variables of the first-order system. Uniqueness patches them at L . The conditions $q(0) = J(0) = 0$ give a smooth even extension, while positivity makes its surface of revolution a regular embedded graph. $\square$

Lemma 3.7 (Validated compactified tail box). *Let $L = 20$, $x = \zeta^{4/3}$, $x_0 = L^{4/3}$ and $p = 57/4$. Subtract $U_4 = \alpha\zeta + \sum_{n=1}^4 c_n\zeta^{1-4n}$ and use the equivalent first normal form*

$$\mathbf{Y} = P(I + S/x)Z, \quad \mathbf{Y} = (V, V_x, V_{xx}, V_{xxx})^T, \quad V = U - U_4.$$

Put

$$\kappa = \frac{3}{4}\{2\mu(1 + \alpha^2)^2\}^{1/3}, \quad (\lambda_0, \lambda_s, \lambda_+, \lambda_-) = (0, -\kappa, \kappa e^{i\pi/3}, \kappa e^{-i\pi/3}),$$

let $r_j = (1, \lambda_j, \lambda_j^2, \lambda_j^3)^T$, and set

$$P = (r_0, r_s, r_+, r_-), \quad \Lambda = \text{diag}(\lambda_0, \lambda_s, \lambda_+, \lambda_-).$$

The x^{-1} matrix in this eigenbasis and its diagonal part are

$$\mathcal{B}_1 = \begin{pmatrix} 3/4 & 15/4 & 15/4 & 15/4 \\ -1/4 & -5/4 & -5/4 & -5/4 \\ -1/4 & -5/4 & -5/4 & -5/4 \\ -1/4 & -5/4 & -5/4 & -5/4 \end{pmatrix}, \quad D = \text{diag}(3/4, -5/4, -5/4, -5/4).$$

Define $S_{ii} = 0$ and $S_{ij} = -(\mathcal{B}_1)_{ij}/(\lambda_i - \lambda_j)$ for $i \neq j$. We work on $Z_- = \overline{Z_+}$ with

$$|Z|_c^2 = |Z_0|^2 + |Z_s|^2 + 2|Z_+|^2;$$

for interval evaluation this is realified by

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & 1 & -i \end{pmatrix}, \quad (Z_0, Z_s, \Re Z_+, \Im Z_+)^T \mapsto Z.$$

All norms below are the induced real conjugate-pair norms. Uniformly for

$$1.0370 \leq \alpha \leq 1.0372, \quad 0.03028 \leq \mu \leq 0.03031, \quad x \geq x_0,$$

the exact transformed equation

$$Z_x = (\Lambda + D/x)Z + \mathcal{R}(x)Z + \mathcal{Q}(x, Z) + g(x)$$

has the following precise meaning. If $f_Z[\overline{U}](x, Z)$ denotes the exact transformed vector field obtained after writing $U = \overline{U} + V$ and applying the displayed normal-form changes, then

$$g = f_Z[U_4](x, 0), \quad \mathcal{R} = D_Z f_Z[U_4](x, 0) - (\Lambda + D/x),$$

and

$$\mathcal{Q}(x, Z) = f_Z[U_4](x, Z) - f_Z[U_4](x, 0) - D_Z f_Z[U_4](x, 0)Z.$$

We write $\mathcal{R}_{\text{cone}} = D_Z f_Z[\alpha\zeta](x, 0) - (\Lambda + D/x)$. With these definitions the transformed equation satisfies, in the real conjugate-pair norm,

$$\|\mathcal{R}(x)\| \leq 100x^{-2}, \quad |g(x)| \leq 4 \times 10^{14}x^{-p-1}, \quad (3.24)$$

$$|\mathcal{Q}(x, Z) - \mathcal{Q}(x, \tilde{Z})| \leq 20x^{1/4}(|Z| + |\tilde{Z}|)|Z - \tilde{Z}|. \quad (3.25)$$

Put

$$\Phi_s(x) = e^{-\kappa(x-x_0)}(x/x_0)^{-5/4}, \quad Z = C_L \Phi_s e_s + Z_r.$$

The stable Volterra component is normalised at the finite base by

$$(Z_r)_s(x_0) = 0, \quad Z_s(x_0) = C_L. \quad (3.26)$$

On $|C_L| \leq 10^{-11}$ and $\|Z_r\|_p := \sup_{x \geq x_0} x^p |Z_r(x)| \leq 10^{14}$, the finite-base Volterra map has Lipschitz constant less than 0.142 and maps the ball into a concentric ball of relative radius less than 0.524. Its unique fixed point depends C^1 on (α, μ, C_L) and satisfies

$$\max\{|Z_0(x_0)|, |Z_+(x_0)|, |Z_-(x_0)|\} \leq 10^{14} 20^{-19} = 1.9073486328125 \times 10^{-11}. \quad (3.27)$$

The entire validation tube is positive; in fact $U(\zeta)/\zeta > 1.0369$ for $\zeta \geq 20$.

Proof. The algebraic recursion cancels the exact compactified residual through s^{18} , $s = \zeta^{-1}$; symbolic rational arithmetic verifies that its first possible power is s^{19} . Outward-rounded evaluation on $s \in [0, 0.05]$ gives

$$\sup_{x \geq x_0} x^{p+1} |g(x)| < 3.101931095 \times 10^{14} < 4 \times 10^{14}.$$

For the remaining bounds put $w = \zeta^{-1/3} = x^{-1/4}$. A 256-bit Arb calculation on $1024 \times 4 \times 4$ boxes covers the rational enlargement

$$0 \leq w \leq 0.3685, \quad 1.0370 \leq \alpha \leq 1.0372, \quad 0.03028 \leq \mu \leq 0.03031.$$

The same boxes validate the interval inverse of $P(I + Sw^4)C$ throughout; equivalently $I + S/x$ is invertible on the whole parameter-tail box. It proves

$$\sup x^2 \|\mathcal{R}_{\text{cone}}\| < 27.406732$$

and evaluates the weighted Hessian separately on the exact cone-to- U_4 segment. On that segment

$$\sup x^{-1/4} \|D_Z^2 f_Z\|_F < 8.633983,$$

while the rescaled base displacement is less than 1.92354. The fundamental theorem of calculus therefore gives a correction less than 16.596376, and hence

$$\sup x^2 \|\mathcal{R}\| < 44.003107 < 100.$$

For the Hessian write $u = w^3 U$, $h_2 = w^2 U''$ and $h_3 = w^3 U'''$. On the full Volterra envelope write $Z = w^{57} E$, thereby defining the bounded rescaled vector E . One has $h_2 = w^{17} b_2$ and $h_3 = w^{21} b_3$, which defines the bounded tube variables b_2, b_3 . After these substitutions all ten entries

$$w^{1+e_a+e_b} \partial_a \partial_b (w^4 U'''), \quad e = (3, -1, 0, 0),$$

with a, b ranging over the ordered variables (u, q, h_2, h_3) , are rational functions regular at $w = 0$. Direct interval composition with $P(I + Sw^4)C$ and its inverse gives

$$\sup x^{-1/4} \|D_Z^2 f_Z[U_4]\|_F < 8.632301 < 20,$$

which implies (3.25) by integral Taylor remainder.

For completeness, put

$$\Gamma_L = \max \left\{ \frac{1}{p + 3/4}, \frac{2}{\kappa x_0} \right\}, \quad \rho = |C_L| + R x_0^{-p}, \quad R = 10^{14}.$$

The interval parameter box gives $\Gamma_L < 0.08$ and $\kappa x_0 > 26.01517 > 2(p - 5/4) = 26$. Thus the stable free orbit is bounded by the same x^{-p} envelope and the forward stable kernel has the factor $2/(\kappa x_0)$ used below. Let $\mathcal{G}$ denote the block solution operator formed from the centre, stable and two unstable Volterra kernels. They satisfy

$$\|\mathcal{G}H\|_p \leq \Gamma_L \sup_{x \geq x_0} x^{p+1} |H(x)|.$$

With $K_2 = 100$, $K_g = 4 \times 10^{14}$ and $K_N = 20$, the Lipschitz and ball ratios are bounded respectively by

$$L_{\text{tail}} \leq \Gamma_L \left\{ \frac{K_2}{x_0} + 2K_N x_0^{5/4} \rho \right\},$$

$$B_{\text{tail}} \leq \frac{\Gamma_L}{R} \left\{ K_g + K_2 |C_L| x_0^{p-1} + K_N x_0^{p+5/4} \rho^2 \right\} + \frac{\Gamma_L K_2}{x_0}.$$

Substitution of these values and $|C_L| \leq 10^{-11}$ gives upper bounds 0.142 and 0.524. The same interval boxes give the stated positive lower bound. Uniform parameter-dependent contraction gives the asserted C^1 endpoint map. Since $t = \kappa x$, the algebraic rescaling and the two normal-form changes are invertible on the validation box. Thus this is the same finite-base tail chart as in [Proposition 3.5](#), expressed in the cancellation-free P_4 coordinates used by the interval calculation. $\square$

Lemma 3.8 (Validated centre cube and signed faces). *Put*

$$p_0 = (0.63169185949612983, 0.030292801498271463, 1.037079401503446)$$

and let $F = (Z_0, \Re Z_+, \Im Z_+)$ be the centre solution in the P_4 finite-base coordinates at $L = 20$. Use the following fixed decimal rational preconditioner:

$$B = \begin{pmatrix} -1.9874531069408703921803188384096016152 \cdot 10^{-8} & -3.89919252084496093216178406302010914955 \cdot 10^{-5} & 2.692876593501873590740705252497522410696 \cdot 10^{-4} \\ 9.93565749626417479846462113618818936 \cdot 10^{-11} & 2.812547048420073565711689937996722092401 \cdot 10^{-5} & -1.724902102072632493014507961965114922251 \cdot 10^{-5} \\ -5.0000228133190469535136000176975175576826 \cdot 10^{-2} & -6.146379043385490350935388358263219194 \cdot 10^{-4} & 1.2908993723726045408121737752290791486 \cdot 10^{-3} \end{pmatrix}.$$

For $p = (\gamma, \mu, \alpha)$ put

$$C_L(p) = Z_{s,c}(20; p)$$

and let $F_\infty(p)$ be the three nonstable coordinates $(Z_0, \Re Z_+, \Im Z_+)$ of the tail endpoint with parameters $(\alpha, \mu, C_L(p))$. The exact nonstable mismatch is

$$M(p) = F(p) - F_\infty(p). \quad (3.28)$$

For $r = 3 \times 10^{-11}$ put

$$p(\xi) = p_0 + rB\xi, \quad \mathcal{B} = p([-1, 1]^3).$$

the physical half-widths in (γ, μ, α) are

$$(9.248983772691192 \times 10^{-15}, 1.361237725845061 \times 10^{-15}, 1.557172962297049 \times 10^{-12}),$$

and the validated centre enclosures satisfy

$$\|F(p_0)\|_\infty < 4.913763 \times 10^{-12}, \quad (3.29)$$

$$\inf_{p \in \mathcal{B}, 0 \leq \zeta \leq 20} U_c(\zeta; p) > 0.9996, \quad (3.30)$$

$$\sup_{p \in \mathcal{B}} |Z_{s,c}(20; p)| < 1.774939 \times 10^{-12}. \quad (3.31)$$

On every face $\xi_i = \sigma$, $\sigma \in \{-1, 1\}$, direct interval evaluation of the exact mismatch satisfies

$$\sigma M_i(p(\xi)) > 6.01275 \times 10^{-12}. \quad (3.32)$$

Proof. A 256-bit Arb calculation retains ξ_1, ξ_2, ξ_3 as global degree-three Taylor-model variables [6] and divides $[0, 20]$ into 400 equal slabs. On each slab it constructs a degree-16 time polynomial P for the flux system. The defect enclosure contains both the exactly evaluated finite coefficients of $P' - f(P)$ and the omitted time tail. For the latter, the code shifts P to an arbitrary τ in the slab, encloses the normalised seventeenth coefficient of $f(P(\tau + s))$, and multiplies it by h^{17} , $h = 1/20$, as required by Taylor's theorem. The constant and three linear parameter noises are retained in this shifted calculation; all higher parameter monomials and the Taylor-model remainder are placed in a symmetric outward-rounded affine error.

Let E^0 be the incoming error, R the resulting residual bound and A a componentwise enclosure of $|D_Y f|$ on the proposed slab tube. Before the endpoint is accepted the program verifies, component by component, the strict inclusion

$$E_i > E_i^0 + h \left(R_i + \sum_j A_{ij} E_j \right).$$

A first-exit argument therefore encloses the exact ODE solution on the whole slab; no unweighted contraction surrogate is used. The lower bounds for U are checked on these same tubes. Propagating the midpoint-selection radius at $t = 0$ and the endpoint rounding radius gives (3.29)–(3.31).

Finally the code evaluates the exact flux-to-jet and $P(I + S/x)$ coordinate maps in Taylor-model arithmetic. On each of the six faces it ranges every remaining monomial, includes the global remainder, and subtracts the symmetric tail endpoint box of radius $10^{14}20^{-19} = 1.9073486328125 \times 10^{-11}$ from (3.27). The six resulting signed lower endpoints are strictly positive, which is (3.32). $\square$

Theorem 3.9 (Certified positive conical profile). *There is a smooth positive even solution of (2.16) with $U(0) = 1$ and conical slope $\alpha > 0$. Writing $\gamma = U''(0)$, its parameters lie in the validated enclosure*

$$\begin{aligned} |\gamma - 0.63169185949612983| &\leq 9.248983772691192 \times 10^{-15}, \\ |\mu - 0.030292801498271463| &\leq 1.361237725845061 \times 10^{-15}, \\ |\alpha - 1.037079401503446| &\leq 1.557172962297049 \times 10^{-12}. \end{aligned}$$

In particular U has the differentiated conical expansion in Proposition 3.5 and is bounded away from zero.

Proof. We verify the zero required by Proposition 3.6 in the validated affine box. For each $p \in \mathcal{B}$, set the stable tail coordinate equal to $C_L = Z_{s,c}(20; p)$. The last bound in Lemma 3.8 places it in the tail chart of Lemma 3.7. The latter supplies a continuous exact tail endpoint and the displacement bound (3.27). The signed-face conclusion of Lemma 3.8 and the Poincaré–Miranda theorem therefore give a zero of the three nonstable matching conditions. The stable condition holds by the definition of C_L . Invertibility of the finite-base coordinate changes gives equality of the four profile jets. On $U > 0$ the exact map from (U, U', U'', U''') to (U, q, K, J) in (3.5)–(3.6) is analytic and invertible, so all four flux variables agree as well. Uniqueness of the first-order system patches the centre and tail solutions; their validated lower bounds give positivity, and the centre conditions give the smooth even extension. $\square$

Remark 3.10 (Validation record). The two interval lemmas above isolate every computer-assisted step. The accompanying deterministic archive is [conical_pincher_certificate.zip](#), also included in the ancillary files in the arXiv submission, with SHA-256

ab18a762fc0257ceae225ca1dc10f15b9d80e6065b1a80040d394fc4992ae2d5

It contains the exact symbolic code, the 256-bit Arb generators, all 400 centre-step records, this numerical supplement and a canonical receipt `certificate_receipt.json`. The receipt has SHA-256

80bc0f112cb3304d1d9259282446f70d3bc285be16c27fdbd3364f8dcb5bdedd

and binds the proof-facing source manifest with aggregate digest

449b5a0c154954cca2a2478d495457955b305485b7027f280c8c396fa6159812

The frozen replay used CPython 3.12.13, python-flint 0.8.0 and SymPy 1.13.3. It recomputed the exact tail recursion, all $1024 \times 4 \times 4$ tail boxes and the 400 order-16 centre slabs, then rehashed the source before writing status `proved`. The standard-library checker `verify_receipt.py`, the sibling receipt digest, the archive digest and every internal archive-member hash all pass. No binary floating-point number is a proof input. The proof uses no numerical eigenvalue count and no floating-point shooting residual. Its foundational ingredients are interval analysis and interval Newton methods [35], the Krawczyk operator [25], the Poincaré–Miranda degree theorem [34], high-order Taylor models [6], and arbitrary-precision ball arithmetic [22]; see also [28, 43] for modern validated ODE and connecting-orbit frameworks.

4 Weighted terminal dynamics and mode selection

The formal similarity linearisation, its geometric translation and dilation modes, and its far-field WKB branches were identified for the fundamental profile in [5]. We put that analysis in the present dynamic-neck normalisation and prove the weighted Fredholm and essential-spectrum statements needed for compactification. The broader methodological lineage includes weighted spectral norms for convective stability [41, 45], exponential-dichotomy and weighted Fredholm theory [29, 30, 38], and Lyapunov–Perron invariant-manifold constructions [20, 4, 45]. The estimates specific to the surface-diffusion operator are proved below.

Fix the profile U from [Theorem 3.9](#). In the following S , K and M are the coefficient functions associated with this fixed profile.

Set

$$S = (1 + U'^2)^{1/2}, \quad K = \frac{1}{US} - \frac{U''}{S^3}, \quad M = \frac{U}{S}.$$

For a perturbation v , put

$$k_U[v] = -\frac{v}{U^2S} - \frac{U'v'}{US^3} - \frac{v''}{S^3} + \frac{3U''U'v'}{S^5}, \quad (4.1)$$

$$m_U[v] = \frac{v}{S} - \frac{UU'v'}{S^3}. \quad (4.2)$$

Direct differentiation of (2.12) gives

$$\mathcal{A}_U v := D\mathcal{F}[U]v = -\frac{v}{2U^2}(MK')' + \frac{1}{2U}(m_U[v]K' + M(k_U[v]))'. \quad (4.3)$$

Its principal part is

$$-\frac{1}{2(1 + U'^2)^2}v^{(4)}. \quad (4.4)$$

On an unrestricted smooth weighted core define

$$\mathcal{L}_0 v = \mathcal{A}_U v + \mu(v - \zeta v'). \quad (4.5)$$

This operator preserves parity. On its even trace slice $v(0) = 0$, define

$$\mathcal{L} v = \mathcal{A}_U v - (\mathcal{A}_U v)(0)(U - \zeta U') + \mu(v - \zeta v') \quad (4.6)$$

If

$$DU = U - \zeta U', \quad \Pi w = w - w(0)DU,$$

then $DU(0) = 1$, $\Pi^2 = \Pi$, and

$$\mathcal{L} = \Pi \mathcal{L}_0 \quad \text{on } \{v : v(0) = 0\}. \quad (4.7)$$

The two geometric modes are exact:

$$\mathcal{L}_0 U' = \mu U', \quad \mathcal{L}_0(DU) = 4\mu DU. \quad (4.8)$$

These are the translation and dilation modes found in the similarity analysis of [5, Section 5.1.1]. Reflection symmetry and the dynamic neck normalisation determine how they enter the present phase space. Reflection symmetry removes the odd translation mode, while dynamic neck normalisation removes DU . The latter operation does not rule out a second eigenvector or a Jordan chain at 4μ .

Lemma 4.1 (The normalising projection). *If $\mathcal{L}_0 \phi = \lambda \phi$ on the smooth core, then*

$$\mathcal{L}(\Pi \phi) = \lambda \Pi \phi \quad (4.9)$$

for every $\lambda \in \mathbb{C}$. At $\lambda = 4\mu$, the projected vector vanishes only when ϕ is proportional to DU .

Proof. Since $\Pi DU = 0$ and $\mathcal{L}_0 DU = 4\mu DU$,

$$\Pi \mathcal{L}_0 \{\phi - \phi(0)DU\} = \Pi \{\lambda \phi - 4\mu \phi(0)DU\} = \lambda \Pi \phi.$$

This is (4.9). The final statement follows from the definition of Π . $\square$

4.1 The conical end and the correct decay topology

Suppose that

$$U(\zeta) = \alpha |\zeta| + b |\zeta|^{-3} + O(|\zeta|^{-7}), \quad q_\alpha = 1 + \alpha^2,$$

with differentiated asymptotics. Then

$$\begin{aligned} \mathcal{L}_0 v = & -\frac{1}{2q_\alpha^2} v^{(4)} + \mu(v - \zeta v') \\ & + O(|\zeta|^{-1} v''' + |\zeta|^{-2} v'' + |\zeta|^{-3} v' + |\zeta|^{-4} v). \end{aligned} \quad (4.10)$$

The formal eigenvalue equation has the algebraic branch

$$v(\zeta) \sim |\zeta|^{1-\lambda/\mu} \quad (4.11)$$

and three WKB factors

$$\exp(\varrho_j \kappa |\zeta|^{4/3}), \quad \varrho_j^3 = -1, \quad \kappa = \frac{3}{4} (2\mu q_\alpha^2)^{1/3}. \quad (4.12)$$

One WKB branch decays and two grow at the conical end. This algebraic/WKB structure is the one obtained formally in [5, Sections 4.1 and 5.1.2]; the Fredholm threshold and exact essential boundary established below are operator-theoretic conclusions of the present work.

The global relative topology cannot decay exponentially. Indeed, for the constant far operator

$$B = -a_\alpha \partial_\zeta^4 + \mu(1 - \zeta \partial_\zeta), \quad a_\alpha = \frac{1}{2q_\alpha^2},$$

one has the exact semigroup formula

$$(e^{sB} f)(\zeta) = e^{\mu s} \left(e^{-a_\alpha \tau(s) \partial_\zeta^4} f \right) (e^{-\mu s} \zeta), \quad \tau(s) = \frac{1 - e^{-4\mu s}}{4\mu}. \quad (4.13)$$

Thus a localised perturbation is carried to $|\zeta| \asymp e^{\mu s}$. The unprojected formula also exposes a point which is essential for the terminal construction: for generic f ,

$$(e^{sB} f)(0) = e^{\mu s} (e^{-a_\alpha \tau(s) \partial_\zeta^4} f)(0),$$

so a constant-like core component grows. It must be cancelled by the future Duhamel condition on the complete controlled spectral space; neck normalisation removes the exact dilation vector but not an additional eigenvector or Jordan chain. After that cancellation, an outgoing front may remain uniformly small in the global relative and slope components without decaying there. Exponential decay is imposed only in a core-discounted norm and, after all controlled modes have been removed, in every fixed local C^j norm.

For the core put

$$H_m^k = H^k(\mathbb{R}, \langle \zeta \rangle^{-2m} d\zeta), \quad m > \frac{3}{2}. \quad (4.14)$$

Polynomially weighted norms used to move convective essential spectrum have a classical role in stability theory [41, 45]. The threshold is exactly the condition that cone-sized perturbations $v = O(|\zeta|)$ belong to the underlying weighted L^2 space. On the positive end, set $y = \log \zeta$ and

$$w(y) = e^{(1/2-m)y} v(e^y).$$

This identifies the weighted L^2 norm asymptotically with the unweighted $L^2(dy)$ norm: the omitted multiplier is $(1 + e^{-2y})^{-m/2} = 1 + O(e^{-2y})$. Conjugation by the displayed leading multiplier gives

$$1 - \zeta \partial_\zeta \mapsto \frac{3}{2} - m - \partial_y, \quad (4.15)$$

$$\partial_\zeta^4 \mapsto e^{-4y} \prod_{j=0}^3 \left(\partial_y + m - \frac{1}{2} - j \right). \quad (4.16)$$

The exact unitary conjugation adds coefficients of order e^{-2y} and does not change the limiting operator. Every remainder in (4.10) has e^{-4y} differential scale. The limiting end operator is therefore the outward transport operator

$$\mu \left(\frac{3}{2} - m - \partial_y \right),$$

whose spectral right boundary is

$$s_{\text{ess}} = -\mu \left(m - \frac{3}{2} \right) < 0. \quad (4.17)$$

For $k > 1/2$ define the normalised even phase Hilbert space

$$\mathcal{X}_m^k = \{v \in H_m^k : v \text{ even}, v(0) = 0\} \quad (4.18)$$

with the inherited norm. One convenient dense graph domain in this space is

$$\mathcal{D}_{m,k} = \left\{ v \in H_m^{k+4} : \zeta v' \in H_m^k, v \text{ even}, v(0) = 0 \right\}. \quad (4.19)$$

Here $\mathcal{L}_0 v$ has a trace at zero and $DU = 4b|\zeta|^{-3} + O(|\zeta|^{-7})$ belongs to the ambient weighted space; although $DU(0) = 1$, the projection Π maps into (4.18). The calculation (4.15)–(4.17) does not by itself prove a resolvent theorem. The required functional-analytic step is an end-parametrix estimate followed by a compact-core, high-frequency resolvent estimate. One cannot simply call the variable-coefficient part relatively compact: the decaying perturbation of the principal coefficient still multiplies $v^{(4)}$ and is not compact from the graph domain to the phase space. This construction is in the tradition of weighted Fredholm theory on noncompact domains and exponential-dichotomy index formulae [30, 38]; the exact line $-\mu/2$ is specific to the dilation transport here.

Proposition 4.2 (Terminal Fredholm theorem and finite-mode splitting). *Let $m = 2$ and $k \geq 2$. The closure of $\mathcal{L}$ on $\mathcal{X}_2^k$ with domain (4.19) generates a C_0 -semigroup. The pencil*

$$\lambda - \mathcal{L} : \mathcal{D}_{2,k} \longrightarrow \mathcal{X}_2^k$$

is Fredholm of index zero for $\Re \lambda > -\mu/2$, is invertible for all sufficiently large positive real λ , and obeys the high-frequency estimate

$$\|v\|_{\mathcal{X}_2^k} \leq C_\delta \|(\lambda - \mathcal{L})v\|_{\mathcal{X}_2^k} \quad (4.20)$$

whenever

$$\Re \lambda \geq -\frac{\mu}{2} + \delta, \quad |\Im \lambda| \geq R_\delta.$$

Its Fredholm essential spectral bound is exactly $-\mu/2$. Choose

$$0 < \omega < \frac{\mu}{2} \quad (4.21)$$

so that $\{\Re \lambda = -\omega\}$ contains no eigenvalue. Then the spectrum in $\{\Re \lambda > -\omega\}$ consists of finitely many eigenvalues of finite algebraic multiplicity. Let P_c be the sum of their Riesz projections, put $E_c = \text{ran } P_c$, and let $E_s = \ker P_c$. Then E_c is finite-dimensional, E_s is invariant, and its semigroup decays at every rate smaller than ω . In particular, every positive, zero and weakly negative core mode may be placed in the single finite-dimensional controlled space E_c .

We henceforth write

$$\mathcal{L}_c := \mathcal{L}|_{E_c}, \quad \mathcal{L}_s := \mathcal{L}|_{E_s}. \quad (4.22)$$

Proof. The generation, Fredholm, high-frequency and essential-spectrum assertions are proved in Section A. Analytic Fredholm theory then makes the spectrum to the right of $-\omega$ discrete with finite algebraic multiplicities. Estimate (4.20) confines it to a compact set and gives the finite Riesz splitting. On the complementary Hilbert subspace the resolvent is uniform on every line $\Re \lambda = -\omega'$ with $0 < \omega' < \omega$; the Gearhart–Prüss theorem [16, 39], applied after shifting by ω' , gives exponential decay. $\square$

In contrast with classical maximal-regularity theory [13, 12], the drift prevents the use of ordinary sectorial maximal regularity. In fact the essential spectrum contains $-\mu/2 + i\mathbb{R}$. If the stable restriction had ordinary L^2 maximal regularity after a shift by $0 < \eta < \omega$, its resolvent would satisfy

$$\sup_{\tau \in \mathbb{R}} \left\| \tau \{i\tau - (\mathcal{L}_s + \eta)\}^{-1} \right\| < \infty.$$

The distance from $i\tau$ to the shifted essential line is the constant $\mu/2 - \eta$, so the resolvent norm is bounded below by $(\mu/2 - \eta)^{-1}$ and the displayed supremum diverges. The correct derivative is the derivative along the dilation characteristics,

$$\mathfrak{D}_\mu v := v_s + \mu \zeta v_\zeta - \mu v. \quad (4.23)$$

It is intrinsic to the shrinking core: if $h_A(z) = A(s)v(z/A(s), s)$ and $A_s = -\mu A$, then

$$\partial_s h_A(z) = A \mathfrak{D}_\mu v(z/A, s). \quad (4.24)$$

For $k \geq 2$ put

$$X_k := \mathcal{X}_2^{k-2}, \quad Y_k := H_2^{k+2}(\mathbb{R})_{\text{even}} \cap \{v(0) = 0\}, \quad \mathcal{T}_k := \mathcal{X}_2^k. \quad (4.25)$$

Thus $\mathcal{T}_k = (X_k, Y_k)_{1/2,2}$ is the natural trace space for the straightened parabolic problem. Notice that Y_k does not contain the drift graph condition: the latter belongs to the stationary generator, whereas the evolution estimate uses the material derivative (4.23).

Proposition 4.3 (Convective maximal regularity). *Let $k \geq 8$. On every interval $I_n = [n, n+1]$, the equation*

$$v_s - \mathcal{L}v = f \quad (4.26)$$

with $f \in L^2(I_n; X_k)$ has the uniform estimate

$$\begin{aligned} \|v\|_{C(I_n; \mathcal{T}_k)} + \|v\|_{L^2(I_n; H_2^{k+2})} + \|\mathfrak{D}_\mu v\|_{L^2(I_n; H_2^{k-2})} \\ \leq C \left(\|v(n)\|_{\mathcal{T}_k} + \|f\|_{L^2(I_n; X_k)} \right), \end{aligned} \quad (4.27)$$

where C is independent of n . Equation (4.26) is understood as

$$\mathfrak{D}_\mu v - \mathcal{A}_U v + (\mathcal{A}_U v)(0)DU = f. \quad (4.28)$$

Let P_c and P_s be the spectral projections in Proposition 4.2, and put

$$\gamma_c := \max \left\{ 0, \max_{\lambda \in \sigma(\mathcal{L}_c)} (-\Re \lambda) \right\}.$$

Here and below the inner maximum is omitted when $E_c = \{0\}$. If $\gamma_c < \eta < \omega$, $e^{\eta s} f \in L^2(\mathbb{R}_+; X_k)$ and $\xi \in P_s \mathcal{T}_k$, there is a unique solution with

$$P_s v(\cdot, 0) = \xi, \quad P_c v(s) = - \int_s^\infty e^{(s-\tau)\mathcal{L}_c} P_c f(\tau) \, d\tau. \quad (4.29)$$

It satisfies the exponentially weighted version of (4.27) on $\mathbb{R}_+$, together with

$$\sup_{s \geq 0} e^{\eta s} \|v(s)\|_{\mathcal{T}_k} \leq C_\eta \left(\|\xi\|_{\mathcal{T}_k} + \|e^{\eta s} f\|_{L^2(\mathbb{R}_+; X_k)} \right). \quad (4.30)$$

The Riesz projections obtained on H_2^k and on X_k agree on their common trace space.

Proof. It is enough first to treat the unnormalised operator $\mathcal{L}_0$. On I_n write $s = n + \theta$ and set

$$y = e^{-\mu\theta} \zeta, \quad w(y, \theta) = e^{-\mu\theta} v(e^{\mu\theta} y, n + \theta). \quad (4.31)$$

Then

$$w_\theta = e^{-\mu\theta} \mathfrak{D}_\mu v.$$

The transformed principal coefficient is

$$-\frac{e^{-4\mu\theta}}{2\{1 + U'(e^{\mu\theta} y)^2\}^2},$$

which is uniformly negative for $0 \leq \theta \leq 1$. All transformed coefficients and their required weighted derivatives are bounded uniformly in n ; the weights $\langle e^{\mu\theta} y \rangle^{-4}$ and $\langle y \rangle^{-4}$ are uniformly equivalent. The ordinary one-dimensional fourth-order parabolic estimate therefore gives

$$\|w\|_{C([0,1]; \mathcal{T}_k)} + \|w\|_{L^2(0,1; H_2^{k+2})} + \|w_\theta\|_{L^2(0,1; H_2^{k-2})} \leq C(\|w(0)\|_{\mathcal{T}_k} + \|\tilde{f}\|_{L^2 X_k}). \quad (4.32)$$

One may obtain this estimate directly by differentiating through order k , integrating the principal term twice by parts, and then applying the same estimate to the equation. The coefficient of w_{yyyy} is separated from zero, so the method of continuity from the constant-coefficient biharmonic equation is closed by the resulting Gårding estimate.

Returning to v gives (4.27). The normalised operator is the quotient of $\mathcal{L}_0$ by $\text{span}\{DU\}$ under the bounded projection Π ; the triangular identity (A.13) therefore transfers the estimate to $\mathcal{L}$.

For the half-line problem it is important not to infer analyticity from the semigroup. Instead let $h_n \in \mathcal{T}_k$ be the terminal trace of the zero-initial-data problem on $[n, n+1]$; in base-space extrapolation notation,

$$h_n = \int_n^{n+1} e^{(n+1-\tau)\mathcal{L}} f(\tau) \, d\tau.$$

The unit-interval estimate controls this trace in $\mathcal{T}_k$. If $x_n = v(n)$, the recurrence $x_{n+1} = e^{\mathcal{L}} x_n + h_n$ is solved by

$$\begin{aligned} P_s x_n &= e^{n\mathcal{L}_s} \xi + \sum_{j=0}^{n-1} e^{(n-1-j)\mathcal{L}_s} P_s h_j, \\ P_c x_n &= - \sum_{j=n}^{\infty} e^{(n-1-j)\mathcal{L}_c} P_c h_j. \end{aligned}$$

Exponential stability on E_s and $\eta < \omega$ control the forward series. On the finite-dimensional space E_c , the strict inequality $\eta > \gamma_c$ controls the backward series, including the polynomial factors from Jordan blocks. Thus both series converge with the required η -weighted trace bound. Solve between consecutive traces by the unit-interval problem and sum (4.27). This proves the half-line estimate without ordinary maximal regularity for the transport-dominated generator. Finally, the resolvent equation bootstraps by four local derivatives, while uniqueness identifies the resolvents on intersections; the contour formula for a Riesz projection proves compatibility across the Sobolev scale. $\square$

Proposition 4.4 (Weighted terminal core semiflow). *Let $k \geq 8$, let $m = 2$, and put*

$$\mathbf{c}_k(v) := \sum_{j=0}^k \sup_{\zeta \in \mathbb{R}} \langle \zeta \rangle^{j-1} |\partial_{\zeta}^j v(\zeta)|. \quad (4.33)$$

We also need a uniformly local conical parabolic graph norm which does not rely on an endpoint smoothing estimate. On $I_n = [n, n+1]$ use the straightening (4.31), put $R_n = R_0 e^{\mu n}$, and cover $|y| \leq 2R_n$ by unit intervals $B_{\rho} = (\rho-2, \rho+2)$. With $Q_{\rho} = B_{\rho} \times [0, 1]$, define

$$\tilde{v}(y, \theta) = e^{-\mu\theta} v(e^{\mu\theta} y, n + \theta), \quad \tilde{f}(y, \theta) = e^{-\mu\theta} f(e^{\mu\theta} y, n + \theta),$$

and put

$$\begin{aligned} \mathfrak{g}_{j,\rho}(v) &:= \|\partial_y^j \tilde{v}\|_{L^{\infty}(0,1;H^2(B_{\rho}))} + \|\partial_y^j \tilde{v}\|_{L^2(0,1;H^4(B_{\rho}))} \\ &\quad + \|\partial_y^j \partial_{\theta} \tilde{v}\|_{L^2(0,1;L^2(B_{\rho}))}. \end{aligned}$$

Then

$$\|v\|_{\mathcal{G}_R^k(I_n)} := \sum_{j=0}^k \sup_{|\rho| \leq 2R_n} \langle \rho \rangle^{j-1} \mathfrak{g}_{j,\rho}(v), \quad (4.34)$$

$$\|f\|_{\mathcal{H}_R^k(I_n)} := \sum_{j=0}^k \sup_{|\rho| \leq 2R_n} \langle \rho \rangle^{j-1} \|\partial_y^j \tilde{f}\|_{L^2(0,1;L^2(B_{\rho}))}. \quad (4.35)$$

The $L^{\infty}H^2$ term is the standard L^2 parabolic trace term. The H^4 derivatives carry the weight of the corresponding source derivative, so this is the graph norm of the fourth-order operator rather than an endpoint smoothing claim. Since $H^2(B_{\rho}) \hookrightarrow C^1(B_{\rho})$ in one dimension, this norm also controls the conical pointwise seminorms used below. Define its trace norm by

$$\|\xi\|_{\text{tr } \mathcal{G}_{R_n}^k} := \inf \left\{ \|V\|_{\mathcal{G}_R^k(I_n)} : V(n) = \xi \right\}. \quad (4.36)$$

We denote by $\text{tr } \mathcal{G}_{R_n}^k$ the Banach space of traces for which this infimum is finite. This trace norm, rather than a C^k seminorm, records the $k+2$ spatial derivatives required at the closed lower time face. We use the same notation after restriction to the moving cylinder and write

$$\|v\|_{\mathbb{C}_R^k} := \sup_{n \geq 0} \|v\|_{\mathcal{G}_R^k(I_n)}, \quad \|f\|_{\mathbb{H}_{R,1}^k} := \sum_{n=0}^{\infty} \|f\|_{\mathcal{H}_R^k(I_n)}.$$

The summation in the source norm is essential: the asymptotic cone-slope channel is neutral under material transport, so a merely bounded forcing on successive characteristic boxes could accumulate linearly in s . and

$$\mathcal{P}^k = \left\{ v \in \mathcal{X}_2^k : \mathfrak{c}_k(v) < \infty \right\}, \quad \|v\|_{\mathcal{P}^k} = \|v\|_{\mathcal{X}_2^k} + \mathfrak{c}_k(v).$$

For complete clarity, define the dilation differential and the nonlinear remainder of the radial operator by

$$\mathcal{D}v := v - \zeta v', \quad \mathcal{R}_F(v) := \mathcal{F}[U + v] - \mathcal{F}[U] - \mathcal{A}_U v.$$

The exact waist-normalised nonlinearity is

$$\mathcal{N}(v) := \mathcal{R}_F(v) - \mathcal{R}_F(v)(0)DU - \{(\mathcal{A}_U v)(0) + \mathcal{R}_F(v)(0)\}\mathcal{D}v, \quad (4.37)$$

and $g \in X_k$ will denote a prescribed additive forcing with zero waist trace. This formula follows directly by substituting $r = A(U + v)$ in (2.7) and choosing the scale speed so that $v(0, s) = 0$. In particular $\mathcal{N}(0) = D\mathcal{N}(0) = 0$, and the displayed forced equation in the proof below is exact, not a formal model. On a sufficiently small ball in $\mathcal{P}^k$ the normalised terminal equation defines a C^1 local semiflow through time one. Its time-one map has the form

$$\Psi(v) = e^{\mathcal{L}}v + \mathcal{R}(v), \quad \mathcal{R}(0) = D\mathcal{R}(0) = 0, \quad (4.38)$$

and, after reducing the ball if necessary,

$$\|\mathcal{R}(v) - \mathcal{R}(w)\|_{\mathcal{P}^k} \leq C(\|v\|_{\mathcal{P}^k} + \|w\|_{\mathcal{P}^k})\|v - w\|_{\mathcal{P}^k}. \quad (4.39)$$

The corresponding forced estimate is uniform on every unit time interval from $L^2 H_2^{k-2} \cap \mathcal{H}_R^k$ to the Hilbert graph norm together with $\mathcal{G}_R^k$. For a half-line solution of the linearised equation $\mathcal{P}_i v = f$, the characteristic boxes also satisfy

$$\|v\|_{\mathbb{C}_R^k} \leq C \left\{ \|v(0)\|_{\text{tr } \mathcal{G}_{R_0}^k} + \sup_{s \geq 0} e^{\eta s} \|v(s)\|_{H_2^k} + \|f\|_{\mathbb{H}_{R,1}^k} \right\}. \quad (4.40)$$

The same assertion holds, with constants independent of $R_0 \geq 1$, when $\mathfrak{c}_k$ is restricted to $|\zeta| \leq 2R_0 e^{\mu s}$ and the nonlinear term is cut off at that moving radius.

Proof. Write the perturbation equation in the equivalent convective form

$$\mathfrak{D}_\mu v - \mathcal{A}_U v + (\mathcal{A}_U v)(0)DU = \mathcal{N}(v) + g.$$

On a unit interval $s = n + \theta$, remove the dilation by setting

$$y = e^{-\mu\theta} \zeta, \quad w(y, \theta) = e^{-\mu\theta} v(e^{\mu\theta} y, n + \theta).$$

The principal fourth-order coefficient becomes

$$-\frac{e^{-4\mu\theta}}{2\{1 + (U' + v')^2\}^2}.$$

It is uniformly negative for $0 \leq \theta \leq 1$ on a small $\mathbf{c}_k$ -ball. With $\varpi = \langle y \rangle^{-4}$, commute through $k - 2$ derivatives and use the $H^{k-2} - H^{k+2}$ duality pairing; after two integrations by parts and interpolation one obtains

$$\begin{aligned} \frac{1}{2} \frac{d}{d\theta} \|w\|_{H_{\varpi}^k}^2 + c \|w\|_{H_{\varpi}^{k+2}}^2 &\leq C \|w\|_{H_{\varpi}^k}^2 + C \mathbf{c}_k(v) \|w\|_{H_{\varpi}^{k+2}}^2 \\ &\quad + C \|g\|_{H_{\varpi}^{k-2}}^2. \end{aligned} \quad (4.41)$$

All derivatives of the weight divided by the weight are bounded. The highest nonlinear commutators have the second form on the right of (4.41); the one-dimensional Moser estimates give no derivative loss. The middle term in (4.41) is absorbed on a small ball. The waist normalisation contributes the nonlocal term $-(\mathcal{A}_U v)(0)DU$ and its nonlinear analogue. Although its trace contains $v^{(4)}(0)$, for $k \geq 8$ the local trace estimate and interpolation give

$$|(\mathcal{A}_U v)(0)| \|DU\|_{H_2^k} \|v\|_{H_2^k} \leq \varepsilon \|v\|_{H_2^{k+2}}^2 + C_\varepsilon \|v\|_{H_2^k}^2,$$

and the nonlinear trace has the same bound with the small factor $\mathbf{c}_k(v)$.

Use a whole-line Friedrichs regularisation together with cutoff weights. The displayed estimate, its time-differentiated form and the one-dimensional tame product inequalities are uniform in the regularising parameter and cutoff radius. Compactness on bounded intervals and weak compactness in the weighted graph norms give a solution on $\mathbb{R}$ in the limit; the difference estimate gives uniqueness and C^1 dependence. The waist projection is retained throughout this approximation, so no artificial finite-interval boundary condition is introduced.

For the non-discounted component, apply interior and initial-boundary L^2 maximal regularity on every Q_ρ after the same dilation straightening. On a unit box the polynomial weights are uniformly comparable, the conical coefficient derivatives have the precise powers needed to commute the weights, and neighbouring boxes have uniformly finite overlap. The constant-coefficient estimate, coefficient freezing and absorption give

$$\|v\|_{\mathcal{G}_R^k(I_n)} \leq C \left\{ \|v(n)\|_{\text{tr } \mathcal{G}_{R_n}^k} + \|v\|_{C(I_n; H_2^k)} + \|g\|_{\mathcal{H}_R^k(I_n)} \right\}. \quad (4.42)$$

The initial half-box estimate supplies precisely the displayed trace term. A partition into the compact core and the unit end boxes, followed by the method of continuity, proves (4.42) on the whole moving cylinder. This is the required conical propagation estimate; in particular, no $L_t^1 C_x^{k-4} \rightarrow L_t^\infty C_x^k$ endpoint is used. To pass to the half-line, follow each end box along the dilation characteristic. The limiting transport equation acts as the identity on the cone-slope component and is dissipative on the complementary local derivatives. After separating the discounted core at $|\zeta| = e^{an}$, the coefficient difference on the remaining end is $O(e^{-4an})$. Variation of constants therefore gives the discrete bound

$$G_{n+1} \leq (1 + Ce^{-4an})G_n + C \|f\|_{\mathcal{H}_R^k(I_n)} + Ce^{-\eta n} \sup_s e^{\eta s} \|v(s)\|_{H_2^k},$$

where G_n is the transported conical trace norm. Since $\sum_n e^{-4an} < \infty$, the product of the first factors is bounded; summing the remaining terms proves (4.40). This also explains why the ℓ^1 source norm, rather than an ℓ^∞ norm, is the natural topology for the neutral slope channel. Taylor expansion of the smooth rational coefficients about U now gives (4.39). The controlled constant-like component is retained here; it is cancelled only by the future condition in Theorem 4.6. For the moving-radius version, the unit interval change of variables sends $R_0 e^{\mu(n+\theta)}$ to the fixed y -radius $R_0 e^{\mu n}$. Conically scaled cutoff derivatives have uniform bounds, so the same energy, difference and tame estimates are independent of R_0 . $\square$

Lemma 4.5 (Controlled-mode localisation). *Let P_c be the finite Riesz projection in Proposition 4.2, let $R(s) = R_0 e^{\mu s}$, and let $\chi_R(\zeta) = \chi(\zeta/R)$ equal one on $|\zeta| \leq R$ and zero on $|\zeta| \geq 2R$. Define*

$$\mathfrak{c}_{k,R}(v; s) = \sum_{j=0}^k \sup_{|\zeta| \leq 2R(s)} \langle \zeta \rangle^{j-1} |\partial_\zeta^j v(\zeta, s)|.$$

If

$$\sup_{s \geq 0} e^{\eta s} \|v(s)\|_{H_2^k} + \left\{ \int_0^\infty e^{2\eta s} \|v(s)\|_{H_2^{k+2}}^2 ds \right\}^{1/2} + \sup_{s \geq 0} \mathfrak{c}_{k,R}(v; s) \leq M$$

with $\omega/2 < \eta < \omega$, then the cutoff nonlinear increment satisfies

$$\|P_c \Pi\{\chi_{R(s)} \mathcal{N}(v(s))\}\|_{L^2(n, n+1; H_2^k)} \leq CM^2 e^{-\sigma_N n}, \quad \sigma_N > \omega, \quad (4.43)$$

after reducing M if necessary. The same estimate holds for differences, with one factor M .

Proof. Realise first the unnormalised operator on $\tilde{X}^0 = H_2^0(\mathbb{R})_{\text{even}}$ and pass to the invariant quotient $\tilde{X}^0 / \text{span}\{DU\}$ described in (A.13). The dual representatives of its Riesz projection satisfy the genuine weighted L^2 adjoint equation. They have no algebraic admissible branch on either outward-transport end. More explicitly, the algebraic solution of the adjoint limiting equation in H_2^0 is

$$\psi_a(\zeta) \sim |\zeta|^{2+\bar{\lambda}/\mu}.$$

Its weighted square is asymptotic to $|\zeta|^{2\text{Re } \lambda/\mu} d\zeta$, and is integrable only when $\text{Re } \lambda < -\mu/2$. Every eigenvalue represented in P_c lies strictly to the right of that line, so this branch is excluded. Since its range is finite-dimensional and consists of smooth generalised eigenvectors, the same dual representation makes $P_c : H_2^{k-2} \rightarrow H_2^k$ bounded. The end dichotomy in Lemma A.1, applied to the adjoint system, therefore gives

$$|\partial_\zeta^j \psi(\zeta)| \leq C_j e^{-c|\zeta|^{4/3}}$$

for every adjoint eigenvector and generalised eigenvector occurring in P_c . The same estimate for a Jordan chain follows inductively from the inhomogeneous adjoint dichotomy. In the zero-trace realisation the adjoint of the waist functional contributes only a distribution at $\zeta = 0$; it therefore does not change either end alternative. Sobolev compatibility of the quotient resolvents, proved in Proposition 4.3, transfers these dual representatives to X_k . Choose

$$0 < a < \min \left\{ \mu, \frac{2\eta - \omega}{4} \right\}$$

and split the nonlinear term at $|\zeta| = e^{an}$. On the inner part, conversion from H_2^k to an unweighted local Sobolev norm costs at most e^{2an} . The local Moser estimate and the unit-interval graph norm thus give the rate $e^{-(2\eta-4a)n}$. On the outer part the displayed adjoint bound dominates every polynomial loss from the conical seminorm and from the moving cutoff. Hence (4.43) holds with any $\omega < \sigma_N < 2\eta - 4a$. The polarised calculation proves the difference estimate. Applying Π only adds the bounded rank-one term $-(\chi_R \mathcal{N}(v))(0)DU$, whose controlled projection satisfies the same quadratic bound. $\square$

The future-integral construction is a Lyapunov–Perron stable-manifold argument [20, 4, 45], adapted here to the convective graph norm and compactly supported mode selection.

Theorem 4.6 (Forced strong-stable graph). *Let $k \geq 8$, take the splitting in Proposition 4.2, and suppose that the normalised inner perturbation satisfies*

$$\mathfrak{D}_\mu v - \mathcal{A}_U v + (\mathcal{A}_U v)(0)DU = \Pi\{\chi_{R(s)} \mathcal{N}(v)\} + E_A(s), \quad (4.44)$$

where $\mathcal{N}(0) = D\mathcal{N}(0) = 0$ and

$$E_A(s) \in X_k, \quad \|E_A(s)\|_{X_k} \leq C_A e^{-\sigma s} \quad \text{for some } \sigma > \omega, \quad \|E_A\|_{\mathbb{H}_{R,1}^k} < \infty.$$

Augment the solution topology by the moving conical graph norm $\mathbb{C}_R^k$, without imposing decay in this component. The spectral construction is performed in H_2^k ; no assertion is made that the individual Riesz projections preserve a global conical space. Let P_c and P_s denote the two spectral projections. For every

$$\frac{\omega}{2} < \eta < \omega$$

if C_A and the stable datum $\xi \in E_s$ are sufficiently small, there is a unique small solution which decays like $e^{-\eta s}$ in the core norm and satisfies

$$\begin{aligned} v(s) = & e^{s\mathcal{L}_s} \xi + \int_0^s e^{(s-\tau)\mathcal{L}_s} P_s \{ \Pi(\chi_{R(\tau)} \mathcal{N}(v)) + E_A \}(\tau) \, d\tau \\ & - \int_s^\infty e^{(s-\tau)\mathcal{L}_c} P_c \{ \Pi(\chi_{R(\tau)} \mathcal{N}(v)) + E_A \}(\tau) \, d\tau. \end{aligned} \quad (4.45)$$

Its initial controlled component is the graph

$$P_c v(\cdot, 0) = - \int_0^\infty e^{-\tau\mathcal{L}_c} P_c \{ \Pi(\chi_{R(\tau)} \mathcal{N}(v)) + E_A \}(\tau) \, d\tau =: \Phi_A(\xi). \quad (4.46)$$

If the full initial graph datum and the forcing are small in the moving conical topology, then the full summed trajectory remains uniformly small in $\mathbb{C}_R^k$. This compatibility is automatic for the compactly supported selection in [Lemma 4.7](#). Decay is asserted only in the discounted core and fixed local topologies.

Proof. On E_s the forward semigroup decays at rate ω , while on the finite-dimensional controlled space

$$\|e^{-t\mathcal{L}_c}\| \leq C e^{\omega t}, \quad t \geq 0.$$

The Duhamel identity is interpreted in the base space X_k ; the convective estimate supplies its H_2^k trace. The future integral in (4.45) therefore converges: the imposed forcing decays faster than $e^{-\omega s}$ and [Lemma 4.5](#) supplies the required rate $e^{-\sigma N s}$ for the controlled projection of the quadratic term. The stable projection needs only the ordinary $e^{-\eta s}$ tame bound. Apply the contraction on successive unit intervals using [Proposition 4.4](#); equivalently, use the continuous graph norm associated with (4.41). Estimate (4.39) supplies the quadratic Lipschitz constant, and the future integral cancels every controlled component. Applying the unit-interval estimate to the full sum, with the cutoff boundary moving along the dilation characteristic $R' = \mu R$, and use (4.42) to keep $\mathbb{C}_R^k$ in a small ball. Individual spectral summands need not have this property. No decay in the moving conical component is asserted. $\square$

Lemma 4.7 (Compactly supported mode selection). *Let*

$$\mathcal{Z} = \{ \psi \in C_c^\infty(\mathbb{R}) : \psi \text{ even, } \psi(0) = 0 \}.$$

$\mathcal{Z}$ is dense in $\mathcal{X}_2^k$, and

$$P_c \mathcal{Z} = E_c.$$

Consequently there are $\psi_1, \dots, \psi_N \in \mathcal{Z}$, $N = \dim E_c$, for which $P_c \psi_1, \dots, P_c \psi_N$ form a basis. If g_A is the small initial matching error, the equations

$$P_c \left(g_A + \sum_{j=1}^N a_j \psi_j \right) - \Phi_A \left(P_s \left(g_A + \sum_{j=1}^N a_j \psi_j \right) \right) = 0 \quad (4.47)$$

have a small solution $a = a(A)$ whenever the matching and strong-stable maps are sufficiently small and C^1 .

Proof. Even cutoffs followed by even mollification approximate every $v \in \mathcal{X}_2^k$ in H_2^k . Since $k > 1/2$, the approximating traces converge to $v(0) = 0$. Subtracting the trace times a fixed even compactly supported bump which equals one at zero restores the zero-trace condition without changing the limit. Thus $\mathcal{Z}$ is dense. The image $P_c \mathcal{Z}$ is a dense linear subspace of the finite-dimensional space E_c and is therefore all of E_c . Choose a basis from this image. At the limiting profile the derivative of the left side of (4.47) with respect to a is the invertible matrix with columns $P_c \psi_j$. The implicit function theorem completes the proof. $\square$

The variations in Lemma 4.7 have fixed compact support in similarity variables. For small A their physical supports lie strictly inside the core, so they do not alter the matching annulus. Sufficiently small coefficients preserve positivity and embeddedness. This is why unknown additional unstable modes change the codimension of the construction but not its existence.

5 Fixed-interface compactification

Compactifying a noncompact singularity model and selecting finitely many exceptional modes has important precedents in other geometric flows, including constructions of selected mean-curvature-flow singularities [44, 1], asymptotically conical Ricci-flow singularities [42], and closed mean-curvature flows modelled on prescribed asymptotically conical shrinkers or tangent flows [27, 9]. These works concern different equations and matching mechanisms. The fixed-interface, volume-preserving surface-diffusion construction below is specific to the present problem.

5.1 A moving terminal match

A compact construction cannot be matched to a frozen outer cone. Let $\tau = T - t$ and use the similarity relation

$$\tau = \frac{A^4}{4\mu}. \quad (5.1)$$

Suppose for the moment that the profile has the differentiated expansion

$$U(\zeta) \sim \alpha\zeta + \sum_{n \geq 1} c_n \zeta^{1-4n}. \quad (5.2)$$

Then, at fixed $z > 0$,

$$AU(z/A) \sim \alpha z + \sum_{n \geq 1} (4\mu)^n c_n \tau^n z^{1-4n}. \quad (5.3)$$

This is precisely the singular part of the backward terminal expansion of the outer cap. Indeed, write

$$r_{\text{out}}(z, \tau) = r_0(z) + \sum_{n \geq 1} \tau^n R_n(z), \quad -\partial_\tau r_{\text{out}} = \mathcal{F}[r_{\text{out}}], \quad (5.4)$$

where $r_0(z) = \alpha z$ near the apex. The coefficients are determined recursively by

$$R_{n+1} = -\frac{1}{n+1} [\tau^n] \mathcal{F} \left(\sum_{j=0}^n \tau^j R_j \right), \quad (5.5)$$

and (2.16) gives

$$R_n(z) = (4\mu)^n c_n z^{1-4n}$$

near zero. At first order this identity is exact already from Proposition 3.1:

$$4\mu b = -\frac{1}{2\alpha(1+\alpha^2)}, \quad (5.6)$$

$$\mathcal{F}[\alpha z] = \frac{1}{2\alpha(1+\alpha^2)} z^{-3}, \quad R_1 = -\mathcal{F}[\alpha z] = 4\mu b z^{-3}.$$

Fix an (α, δ) -admissible terminal meridian Γ_0 in the sense of [Definition 1.1](#). Thus the conical germ is fixed on $|z| < 4\delta$, while the outer arc is smooth, embedded, separated from the axis and has a positive tubular radius. For $A > 0$, replace the apex by $AU(z/A)$ and interpolate with (5.4). After reducing A/δ , every resulting meridian is a smooth simple loop in $\{r > 0\}$ and therefore generates an embedded torus. All constants below are uniform when Γ_0 ranges over a sufficiently small admissible C^{k+8} neighbourhood: the perturbation is supported outside $|z| < 4\delta$, and the distance-to-axis, tubular-radius and C^{k+8} bounds are kept uniform.

Denote the terminal rotational surface by Σ_0 and the patched smooth surface at scale A by $\tilde{\Sigma}_A$, with a fixed smooth choice of embedding $\tilde{X}_A$ and unit normal $\tilde{\nu}_A$. For a normal graph $X_A^u = \tilde{X}_A + u\tilde{\nu}_A$, let $\mathcal{S}_A^\sharp(u)$ denote its surface-diffusion normal speed, pulled back to the reference surface, and put

$$\mathcal{T}_A^\sharp(u) := \langle \partial_A X_A^u, \nu[X_A^u] \rangle.$$

Tangential terms are fixed by this reference gauge. Put $\beta_0(A) = -\mu A^{-3}$. In the fixed normal gauge define its invariance defect and signed volume error by

$$\mathcal{Q}_A := \beta_0(A)\mathcal{T}_A^\sharp(0) - \mathcal{S}_A^\sharp(0), \quad \Delta V_A := \text{Vol}(\tilde{\Sigma}_A) - V_0, \quad (5.7)$$

where $V_0 := \text{Vol}(\Sigma_0)$ is the enclosed volume of the terminal surface. An outer normal field with nonzero volume derivative uniquely removes ΔV_A ; the resulting fixed-volume surface and embedding are denoted by $\bar{\Sigma}_A$ and $\bar{X}_A$.

Proposition 5.1 (Fixed-interface matching). *Fix $\delta > 0$. Assume*

$$U(\zeta) = \alpha\zeta + b\zeta^{-3} + c\zeta^{-7} + O(\zeta^{-11})$$

with differentiated asymptotics. Match $AU(z/A)$ on $|z| \asymp \delta$ to the first outer terminal expansion $r_0 + \tau R_1$, where $\tau = A^4/(4\mu)$ and $R_1 = -\mathcal{F}[r_0]$. Then, for every fixed j ,

$$\left| \partial_z^j (r_{\text{in}} - r_{\text{out}}^{(1)}) \right| \leq C_j A^8 \delta^{-7-j}, \quad (5.8)$$

$$\left| \partial_z^j \mathcal{Q}_A \right| \leq C_j A^4 \delta^{-7-j}, \quad (5.9)$$

$$\Delta V_A = O(A^8 \delta^{-5}), \quad (5.10)$$

$$\text{Area}(\bar{\Sigma}_A) = \text{Area}(\Sigma_0) + \mathcal{A}_{\text{ren}}[U]A^2 + O(A^8 \delta^{-6}). \quad (5.11)$$

A fixed outer volume variation has amplitude $O(A^8 \delta^{-5})$ and physical velocity $O(A^4 \delta^{-5})$. After applying this variation, the surfaces $\bar{\Sigma}_A$ form a smooth fixed-volume family and the same defect estimates hold.

If $g_A = A^3 \mathcal{Q}_A$ is written in the similarity variable, then for $m > 3/2$

$$\|g_A\|_{H_m^k} \leq C(A/\delta)^{m+13/2}. \quad (5.12)$$

In particular, for $m = 2$ the source decays like $e^{-17\mu s/2}$.

Proof. The inverse-cubic terms agree because $R_1(z) = 4\mu b z^{-3}$. Thus the first omitted physical term is $cA^8 z^{-7}$, which gives (5.8). Differentiating it along $A_t = -\mu A^{-3}$ gives (5.9); its fourth-order spatial evolution is smaller by $(A/\delta)^4$. Relative-volume neutrality removes the A^3 term. To see the cancellation at order A^4 globally, write

$$\Phi_0 = \frac{r_0}{(1 + r_0^2)^{1/2}} (H_{r_0})_z.$$

The local volume-flux identity gives $r_0 R_1 = -\partial_z \Phi_0$. On the conical side of the annulus,

$$\Phi_0(\delta) = -\frac{1}{2(1 + \alpha^2)\delta} = \frac{4\mu\alpha b}{\delta},$$

while the flux vanishes at the reflected outer endpoint. Consequently the one-sided outer contribution to the $r^2 dz$ moment is

$$2\tau \int_{\delta}^{z_{\text{cap}}} r_0 R_1 dz = \frac{2\alpha b}{\delta} A^4,$$

where z_{cap} is the reflected endpoint of the chosen graphical outer branch. This cancels the term $-2\alpha b A^4/\delta$ in (3.13). Integration of the first unmatched tail now gives (5.10); the relative-area identity gives (5.11).

Put $R = \delta/A$. In similarity variables $|\partial_{\zeta}^j g_A| \leq C_j R^{-7-j}$ for $|\zeta| \gtrsim R$. Hence

$$\|g_A\|_{H_m^k}^2 \leq C \sum_{j=0}^k \int_R^{\infty} \zeta^{-2(7+j)-2m} d\zeta \leq C R^{-2m-13},$$

which proves (5.12). $\square$

5.2 The fixed-interface half-line section

Let $\bar{\Sigma}_A$, $0 < A \leq A_0$, be the fixed-volume family in Proposition 5.1, put $\beta_0(A) = -\mu A^{-3}$, and choose embeddings $\bar{X}_A$ and normals $\bar{\nu}_A$. For $X_A^u = \bar{X}_A + u_A \bar{\nu}_A$, retain the normal-speed and A -variation notation $\mathcal{S}_A^{\sharp}(u)$ and $\mathcal{T}_A^{\sharp}(u)$ introduced above. The exact normal-graph invariance equation is

$$\mathcal{S}_A^{\sharp}(u_A) = \beta(A) \mathcal{T}_A^{\sharp}(u_A). \quad (5.13)$$

Use the half-line time

$$s = \frac{1}{\mu} \log \frac{A_0}{A}, \quad A = A_0 e^{-\mu s}. \quad (5.14)$$

Write

$$\vartheta(s) = A(s)^3 \{\beta(s) - \beta_0(s)\}.$$

The actual physical clock obeys

$$\frac{dt}{ds} = \frac{A(s)^4}{1 - \vartheta(s)/\mu}. \quad (5.15)$$

For the linear splitting use instead the fixed base clock

$$t_*(s) = t_0 + \int_0^s A(\sigma)^4 d\sigma = t_0 + \frac{A_0^4}{4\mu} (1 - e^{-4\mu s}), \quad T_* = t_0 + \frac{A_0^4}{4\mu}. \quad (5.16)$$

The fixed physical interface corresponds to the moving similarity radius

$$R(s) = \frac{\delta}{A(s)} = \frac{\delta}{A_0} e^{\mu s}. \quad (5.17)$$

The clock ratio in (5.15) is retained in the speed block. The cap lies on a fixed smooth physical domain, and

$$\int_0^{\infty} (A/\delta)^4 ds = \frac{(A_0/\delta)^4}{4\mu}. \quad (5.18)$$

To avoid artificial fourth-order boundary conditions, fix a smooth closed double M_o of the physical cap M_o^+ . Choose bounded maps

$$\text{Res}_o : C^{\infty}(M_o) \rightarrow C^{\infty}(M_o^+), \quad \text{Ext}_o : C^{\infty}(M_o^+) \rightarrow C^{\infty}(M_o), \quad \text{Res}_o \text{Ext}_o = I,$$

and put $\Gamma_o = I - \text{Ext}_o \text{Res}_o$. Extend the physical cap linearisation smoothly to a uniformly fourth-order parabolic operator $\hat{G}_*$ on M_o . The equation on the artificial half will be $\Gamma_o(\partial_{t_*} - \hat{G}_*)w = 0$; hence the doubling introduces no uncontrolled ghost degrees of freedom.

Lemma 5.2 (Uniform outer-cap interval). *Fix an admissible cap Γ_0 and $k \geq 10$. There are constants $\rho_{\text{out}}, \tau_{\text{out}} > 0$ and a sufficiently small admissible C^{k+8} neighbourhood $\mathcal{U}$ of Γ_0 such that the extended outer normal-graph equation on M_o is uniformly fourth-order parabolic and has the standard maximal-regularity solution on every interval of physical length at most τ_{out} , for initial graphs and sources in the corresponding ρ_{out} ball. The solution remains inside the fixed normal tube, and the constants are uniform for caps in $\mathcal{U}$.*

In the compactification we impose

$$\frac{A_0^4}{2\mu} < \tau_{\text{out}}. \quad (5.19)$$

After the matching and trace data are reduced as below, the outer component then remains smooth and inside this tube for the whole interval $[t_0, T_)$. In particular, no singularity can form on the outer cap before the inner waist reaches the axis.*

Proof. The positive tubular radius and the C^{k+8} bounds in Definition 1.1 give a fixed C^1 normal-graph ball on which the principal symbol of the extended outer operator is uniformly strongly elliptic and all coefficient maps have uniform tame bounds. Quasilinear fourth-order maximal regularity on the closed manifold M_o therefore gives a positive local existence time and continuous dependence on the cap, initial trace and source. Shrinking the cap neighbourhood, the graph ball and the time once makes these constants uniform and keeps the solution in the normal tube. The base-clock interval in (5.16) has length exactly $A_0^4/(4\mu)$, while the actual clock is bounded by twice this length once $\|\Theta_\beta\|_{L^\infty} < \mu/2$. Condition (5.19), together with the small residual and free-trace bounds in Theorem 5.7, places the entire outer block inside the preceding local theory. Since the reconstructed global surface agrees with that block on the outer region, an earlier outer singularity is excluded. $\square$

Lemma 5.3 (Uniform fixed-volume chart). *There are an even smooth outer normal field $\phi\bar{\nu}_A$, supported away from the matching annulus, and constants $r_0, C > 0$, independent of $0 < A \leq A_0$, with the following property. If h is a symmetric normal graph with $\|h\|_{C^1} < r_0$, there is a unique scalar $\Theta_A(h)$ such that*

$$\text{Vol}(\bar{X}_A + \{h + \Theta_A(h)\phi\}\bar{\nu}_A) = \text{Vol}(\bar{\Sigma}_A). \quad (5.20)$$

The maps are jointly C^2 in $(-\log A, h)$, $\Theta_A(0) = 0$, and

$$|\Theta_A(h)| \leq C\|h\|_{C^1}, \quad \|(A\partial_A)^j D_h^\ell \Theta_A(h)\| \leq C, \quad 1 \leq j + \ell \leq 2, \quad (5.21)$$

on that ball. If h is supported in the inner core, the added variation does not change it.

Proof. Choose ϕ on a fixed compact subset of the outer cap so that $\int \phi \, d\mu_{\Sigma_0} \neq 0$. The approximate caps converge smoothly there as $A \downarrow 0$, and therefore

$$\left| \partial_\theta \Big|_{\theta=0} \text{Vol}\{\bar{X}_A + (h + \theta\phi)\bar{\nu}_A\} \right| \geq c_0 > 0$$

after reducing A_0 and r_0 . The volume functional is smooth in a C^1 normal-graph neighbourhood and jointly smooth in $-\log A$ on this fixed outer set. Differentiating its implicit equation once and twice in $A\partial_A$ and in h , and dividing by the displayed uniformly nonzero derivative, gives (5.20)–(5.21). The support assertion is immediate. $\square$

Lemma 5.4 (Radial-normal gauge conjugacy). *On the graphical core of $\bar{\Sigma}_A$, write the meridian as $\bar{\gamma}_A(z) = (z, \bar{r}_A(z))$. There are neighbourhoods of zero, uniform after measuring length in units of $A + |z|$, and smooth maps*

$$q \longmapsto (\psi_A(q), \mathfrak{N}_A(q))$$

such that

$$(z + \psi_A(q)(z), \bar{r}_A(z + \psi_A(q)(z)) + q(z + \psi_A(q)(z))) = \bar{\gamma}_A(z) + \mathfrak{N}_A(q)(z)\bar{\nu}_A(z). \quad (5.22)$$

At $q = 0$,

$$D\mathfrak{N}_A(0)q = q\langle e_r, \bar{\nu}_A \rangle, \quad D\psi_A(0)q = -q \frac{\bar{r}'_A}{1 + \bar{r}_A^2}. \quad (5.23)$$

Since $\langle e_r, \bar{\nu}_A \rangle$ is uniformly separated from zero on the graphical core, sufficiently small relative C^1 graphs give a one-to-one pointwise chart with inverse, preserving smoothness, and the chart and its inverse have uniform geometric C^1 bounds. For any such small u , write $q = \mathfrak{N}_A^{-1}(u)$. If F is a scalar normal-speed residual on the left side of (5.22), define

$$\mathfrak{R}_A[u]F := \left(\frac{F}{\langle \nu[X_A^u], e_r \rangle} \right) \circ (I + \psi_A(q))^{-1}. \quad (5.24)$$

Then, exactly and not merely to leading order,

$$\mathfrak{R}_A[u]\{\beta\mathcal{T}_A^\sharp(u) - \mathcal{S}_A^\sharp(u)\} = \beta\partial_A(\bar{r}_A + q) - \mathcal{F}[\bar{r}_A + q] \quad (5.25)$$

in the fixed radial coordinate. Higher Banach differentiability of these composition maps is used only with the usual derivative loss; no equal-regularity C^2 assertion is made.

Proof. At each z , equation (5.22) is a system of two scalar equations for (ψ, h) . At $(q, \psi, h) = (0, 0, 0)$ its Jacobian in (ψ, h) has columns $\bar{\gamma}'_A(z)$ and $-\bar{\nu}_A(z)$, whose determinant has absolute value $|\bar{\gamma}'_A|$. The scaled C^1 bounds for the matched family give a uniform lower bound. The parameter-dependent implicit function theorem, followed by differentiation in the scaled variables, gives the chart and its uniform bounds. Projecting the differentiated identity onto the tangent and normal directions gives (5.23). Finally a radial velocity r_t has normal component $r_t\langle e_r, \nu \rangle$; the same identity holds for the A -velocity. Dividing the normal residual by this nonzero factor and undoing the reparametrisation proves (5.25). $\square$

Fix physical cutoffs $\chi_i + \chi_o = 1$, with $\chi_i = 1$ on $|z| \leq \delta/2$ and $\chi_o = 1$ on $|z| \geq 2\delta$. Choose in addition a residual cutoff ρ_i which is one on a neighbourhood of $\text{supp } \chi_i$ (in particular at the waist and on $\text{supp } \nabla \chi_i$) and whose transition to zero lies entirely inside the outer graphical patch. Thus the inner equation uses the actual physical residual throughout the inner-only region and the overlap where the two local graph representatives are identified. Let $M_A = D\mathfrak{N}_A(0)$ be the base radial-normal multiplier from (5.23). The A -dependent local realisations are the linear maps

$$J_i^A v = M_A(Av(\cdot/A, s))(z), \quad J_o^A w = \text{Res}_o w(\cdot, t_*(s)). \quad (5.26)$$

For an inner function v and a doubled-cap function w , set

$$h_A[v, w] = \chi_i J_i^A v + \chi_o J_o^A w, \quad u_A[v, w] = h_A[v, w]. \quad (5.27)$$

The chart in Lemma 5.3 will be used only on the initial slice $A = A_0$ to choose one outer scalar. It is deliberately not inserted slice-by-slice into (5.27): doing so would make the pre-height direction ϕ redundant. For an exact invariant family, volume constancy follows automatically from the equation and is fixed by that single initial scalar. The waist gauge is $v(0, s) = 0$. Let

$$\mathcal{P}_i = \mathfrak{D}_\mu - \mathcal{A}_U + (\mathcal{A}_U \cdot)(0)DU, \quad \mathcal{P}_o = \partial_{t_*} - \widehat{G}_*.$$

If F is a physical scalar residual, let $Q_i^A F$ denote $A^3 \mathfrak{R}_A[0]F(A\zeta)$ on $|A\zeta| < 2\delta$, and let $Q_o^A F$ denote its base-clock pullback to M_o^+ followed by Ext_o . This is a fixed target trivialisation. For

the actual normal height u_A , put $q_A = \mathfrak{N}_A^{-1}(u_A)$ on the core. If the two local representatives agree, then $q_A = Av + O(\|(v, w)\|^2)$, and

$$\mathfrak{R}_A[0]F = \mathfrak{R}_A[0]\mathfrak{R}_A[u_A]^{-1}\{\mathfrak{R}_A[u_A]F\}.$$

For smooth graphs the first factor is algebraically invertible and equals the identity at zero. We use this factorisation only to identify the zero set and the linearisation, not as an equal-regularity composition estimate. Thus the fixed trivialisation has the same zero set as the physical residual and its core linearisation is exactly $\mathcal{P}_i$. Write $\rho_i^A(\zeta) = \rho_i(A\zeta)$. For a provisional speed $\beta = \beta_0 + A^{-3}\vartheta$, define

$$\mathcal{E}_i^{\text{raw}}(v, w, \vartheta) := \rho_i^A Q_i^A \{\beta \mathcal{T}_A^\#(u_A) - \mathcal{S}_A^\#(u_A)\} + (1 - \rho_i^A) \mathcal{P}_i v. \quad (5.28)$$

On $\{\rho_i^A = 1\}$, the preceding algebraic identity and Equation (5.25) show that this term has the same zero set as the scaled radial residual. Differentiating at the similarity profile and the base clock gives $\mathcal{P}_i$; no order-one normal–radial conjugacy is hidden in the cutoff blocks. At $\zeta = 0$ its derivative with respect to ϑ tends to 1, because $\partial_A \bar{r}_A(0) = 1$. The scalar implicit function theorem, uniformly in s , gives a unique

$$\vartheta = \Theta_\beta(v, w), \quad \mathcal{E}_i^{\text{raw}}(v, w, \Theta_\beta(v, w))(0, s) = 0. \quad (5.29)$$

Set $\beta[v, w] = \beta_0 + A^{-3}\Theta_\beta(v, w)$ and define

$$\mathcal{E}_i(v, w) := \Pi \mathcal{E}_i^{\text{raw}}(v, w, \Theta_\beta(v, w)), \quad (5.30)$$

$$\mathcal{E}_o(v, w) := Q_o^A \{\beta[v, w] \mathcal{T}_A^\#(u_A) - \mathcal{S}_A^\#(u_A)\} + \Gamma_o \mathcal{P}_o w. \quad (5.31)$$

Thus $\mathcal{E}_i$ takes values in the zero-waist space X_k , and (5.31) imposes an equation on the whole double. Put

$$F_i(v, w) := \mathcal{P}_i v - \mathcal{E}_i(v, w) \quad (5.32)$$

and define the future boundary functional explicitly by

$$\mathcal{B}_c(v, w) := P_c v(\cdot, 0) + \int_0^\infty e^{-\tau \mathcal{L}_c} P_c F_i(v, w)(\tau) \, d\tau \in E_c. \quad (5.33)$$

The integral is interpreted in X_k and takes values in E_c by controlled-mode localisation. With the choice $\eta > \gamma_c$ made below, the weighted decay built into $\mathfrak{X}$ implies $e^{-s \mathcal{L}_c} P_c v(s) \rightarrow 0$ (including every Jordan factor). Duhamel's formula therefore gives the identity

$$\mathcal{B}_c(v, w) = - \int_0^\infty e^{-\tau \mathcal{L}_c} P_c \mathcal{E}_i(v, w)(\tau) \, d\tau. \quad (5.34)$$

Consequently the future condition is enforced by inverting the core equation on the full weighted space; it is not an additional domain or target constraint. In particular every zero of the equation component automatically satisfies $\mathcal{B}_c = 0$. The free boundary trace and the nonlinear section are defined below, after the compatible trace space has been split off from the controlled directions. Equations (5.29) and (5.31) show explicitly how the waist and artificial-half conditions enter this single map. Volume is imposed by one scalar equation on the initial slice after the dynamical section has been solved.

We now specify the section spaces. Choose

$$\gamma_c := \max \left\{ 0, \max_{\lambda \in \sigma(\mathcal{L}_c)} (-\Re \lambda) \right\} < \omega$$

with the same convention when $E_c = \{0\}$, and then choose

$$\max \left\{ \frac{\omega}{2}, \gamma_c \right\} < \eta < \omega, \quad 0 < a < \min \left\{ \mu, \frac{2\eta - \omega}{4} \right\}, \quad \omega < \sigma_N < \min \left\{ \frac{\mu}{2}, 2\eta - 4a \right\}. \quad (5.35)$$

For the core set

$$\begin{aligned} \|v\|_{\mathbb{I}_\eta} &:= \sup_{s \geq 0} e^{\eta s} \|v(s)\|_{H_2^k} + \left\{ \int_0^\infty e^{2\eta s} (\|v(s)\|_{H_2^{k+2}}^2 + \|\mathfrak{D}_\mu v(s)\|_{H_2^{k-2}}^2) \, ds \right\}^{1/2} \\ &\quad + \|v\|_{\mathbb{C}_R^k} + \|\mathcal{P}_i v\|_{\mathbb{H}_{R,1}^k}, \end{aligned} \quad (5.36)$$

$$\|f_i\|_{\mathbb{Y}_{i,\eta}} := \left\{ \int_0^\infty e^{2\eta s} \|f_i(s)\|_{X_k}^2 \, ds \right\}^{1/2} + \|f_i\|_{\mathbb{H}_{R,1}^k}. \quad (5.37)$$

Let $\mathbb{I}_\eta$ be the space of even waist-normalised paths for which the right side of (5.36) is finite, and let $\mathbb{Y}_{i,\eta}$ be the intersection space defined by (5.37). The last term in (5.36) is the conical graph-domain condition. It excludes an arbitrary nonsummable drift of the neutral cone slope and makes $\mathcal{P}_i : \mathbb{I}_\eta \rightarrow \mathbb{Y}_{i,\eta}$ a well-defined bounded map in the end component. On the closed cap double let

$$E_0 = H^k(M_o), \quad E_1 = H^{k+4}(M_o), \quad E_{1/2} = (E_0, E_1)_{1/2,2} = H^{k+2}(M_o).$$

On $I_* = [t_0, T_*)$ use the standard high-order L^2 maximal-regularity pair

$$\mathbb{O} := H^1(I_*; E_0) \cap L^2(I_*; E_1) \cap C(\bar{I}_*; E_{1/2}), \quad (5.38)$$

$$\mathbb{Y}_o := L^2(I_*; E_0). \quad (5.39)$$

Ordinary compact-manifold maximal regularity supplies the outer inverse with initial trace $E_{1/2}$. No volume constraint is imposed on the Newton domain. At an exact zero, invariance and $\int_\Sigma \mathcal{S} \, d\mu = 0$ imply that the enclosed volume is constant in A ; one scalar on the initial slice fixes its value.

Let $\mathcal{C}_\delta$ be a fixed closed collar in the common graphical domain of the two local charts, containing $\text{supp } \nabla \chi_i$ and contained in $\{\rho_i = 1\}$. Define the path compatibility operator

$$\mathfrak{C}_A(v, w) := (J_i^A v - J_o^A w)|_{\mathcal{C}_\delta} \quad (5.40)$$

and, using the inverse graph-speed multiplier, the source compatibility operator

$$\mathfrak{C}_A^Y(f_i, f_o) := \left\{ \text{Res}_o f_o - (\mathfrak{R}_A[0])^{-1} (A^{-3} f_i(\cdot/A, s)) \right\}|_{\mathcal{C}_\delta}. \quad (5.41)$$

Let $\mathfrak{X}$ be the completion, in the sum norm below, of smooth pairs in $\mathbb{I}_\eta \times \mathbb{O}$ satisfying $\mathfrak{C}_A(v, w) = 0$ for every s . Likewise let $\mathfrak{Y}_{\text{eq}}$ be the completion of smooth source pairs satisfying $\mathfrak{C}_A^Y(f_i, f_o) = 0$ in the sum of the two source norms. These completions avoid any unstated ambient trace or restriction theorem at the terminal time. They are the Banach spaces of two local representatives of one completed global height and one completed global residual. The exact nonlinear chart $\mathfrak{N}_A$ is deliberately absent from their definition; it occurs only in the smooth coefficients of the section.

Let

$$\text{Tr}(v, w) = (v(\cdot, 0), w(t_0)), \quad \mathcal{T}^\# := \text{Tr } \mathfrak{X},$$

where $\mathcal{T}^\#$ is equipped with the quotient norm from $\mathfrak{X}$. The compactly supported functions $\psi_1, \dots, \psi_N$ in Lemma 4.7 give an isomorphism

$$B : \mathbb{R}^N \longrightarrow E_c, \quad Ba = \sum_{j=1}^N a_j P_c \psi_j.$$

Define the compatible controlled lift and the free-trace projection by

$$\mathcal{J}_c c := \left(\sum_{j=1}^N (B^{-1} c)_j \psi_j, 0 \right) \in \mathcal{T}^\#, \quad (5.42)$$

$$\mathcal{P}_f(v_0, w_0) := (v_0, w_0) - \mathcal{J}_c(P_c v_0). \quad (5.43)$$

Thus

$$\mathcal{T}_f^\# := \mathcal{P}_f \mathcal{T}^\# = \{(v_0, w_0) \in \mathcal{T}^\# : P_c v_0 = 0\}, \quad \mathcal{T}^\# = \mathcal{T}_f^\# \oplus \mathcal{J}_c E_c.$$

Finally set $\mathfrak{Y} = \mathfrak{Y}_{\text{eq}} \times \mathcal{T}_f^\#$. For a prescribed free trace $b \in \mathcal{T}_f^\#$ define

$$\mathcal{S}_b(v, w) := \begin{pmatrix} \mathcal{E}_i(v, w) \\ \mathcal{E}_o(v, w) \\ \mathcal{P}_f \text{Tr}(v, w) - b \end{pmatrix}, \quad \mathcal{S} := \mathcal{S}_0. \quad (5.44)$$

On $\mathcal{C}_\delta$ one has $\rho_i = 1$ and $h_A = J_i^A v = J_o^A w$. Hence the two equation components are exactly the two fixed local representatives of the same physical residual, and (5.41) holds identically. It follows that $\mathcal{S}_b$ is a map between these fixed spaces, rather than a section of a moving nonlinear residual bundle. Let $\mathfrak{Z}_\beta$ be the weighted space of continuous scalar functions with the norm below. Explicitly,

$$\begin{aligned} \|(v, w)\|_{\mathfrak{X}} &:= \|v\|_{\mathbb{I}_\eta} + \|w\|_{\mathbb{O}}, \\ \|(f_i, f_o, b)\|_{\mathfrak{Y}} &:= \|f_i\|_{\mathbb{Y}_{i,\eta}} + \|f_o\|_{\mathbb{Y}_o} + \|b\|_{\mathcal{T}^\#}, \end{aligned} \quad (5.45)$$

$$\|\vartheta\|_{\mathfrak{Z}_\beta} := \sup_{s \geq 0} e^{\eta s} |\vartheta(s)|. \quad (5.46)$$

The quotient norm on $\mathcal{T}^\#$ controls both the inner trace norm from (4.36) with $n = 0$ and the outer $E_{1/2}$ trace norm.

The separate inner and outer variables are only local representations of one global height; estimating the four formal product-space blocks individually would therefore count the principal overlap operator twice. The correct linear statement is the following fibre-product estimate.

Lemma 5.5 (Uniform fibre-product inverse). *Put $\varepsilon = A_0/\delta$. For ε sufficiently small, the linearisation*

$$D\mathcal{S}(0) : \mathfrak{X} \longrightarrow \mathfrak{Y}$$

has a bounded inverse on the admissible trace fibre, and

$$\|(v, w)\|_{\mathfrak{X}} + \|D\Theta_\beta(0)(v, w)\|_{\mathfrak{Z}_\beta} \leq C \|D\mathcal{S}(0)(v, w)\|_{\mathfrak{Y}}. \quad (5.47)$$

The constant is independent of $0 < \varepsilon \leq \varepsilon_0$.

Proof. We give the localisation argument because it carries the point lost in an unrestricted product space: physical-time regularity on the overlap.

The boundary component in (5.44) prescribes only the free part of a full compatible trace. If that part is b , write the unknown initial trace uniquely as $b + \mathcal{J}_c c$. In the decoupled core problem the future formula (4.29) determines $c \in E_c$ uniquely from the forcing; its derivative with respect to c is the identity. The finite-dimensional equation remains invertible for small ε by the localisation estimate below. This is the joint initial–future compatibility that would be missed by prescribing the stable and outer traces independently.

On the inner chart, the exact radial–normal identity (5.25) conjugates the principal linearisation to $\mathcal{P}_i$. The unit-interval estimate (4.27), its uniformly local counterpart (4.42), and the future choice on E_c give the inner estimate. The summable estimate (4.40), together with the equation, also controls $\|\mathcal{P}_i v\|_{\mathbb{H}_{R,1}^k}$; the end coefficient errors are summable and are absorbed after decreasing ε_0 . On the cap, ordinary compact-manifold maximal regularity gives the outer estimate; the completed equation controls the artificial half of the double.

On $\mathcal{C}_\delta$ the compatibility equation (5.40) says that the two fields are the same physical normal graph. In particular an arbitrary oscillation in the logarithmic time s is not admissible unless it has the physical $H_{t_*}^1 H^k$ regularity supplied by the outer representative. Localising the common

graph to a slightly larger collar and using (5.41) gives the ordinary estimate for the physical linearisation

$$\mathcal{L}_A^{\text{phys}} := D_u \{ \beta_0(A) \mathcal{T}_A^\#(u) - \mathcal{S}_A^\#(u) \}_{u=0}.$$

$$\begin{aligned} & \|h\|_{H^1(I_*; H^k(\mathcal{C}_\delta))} + \|h\|_{L^2(I_*; H^{k+4}(\mathcal{C}_\delta))} + \|h\|_{C(\bar{I}_*; H^{k+2}(\mathcal{C}_\delta))} \\ & \leq C \left(\|\mathcal{L}_A^{\text{phys}} h\|_{L^2(I_*; H^k(\mathcal{C}'_\delta))} + \|h(t_0)\|_{H^{k+2}(\mathcal{C}'_\delta)} + \|h\|_{L^2(I_*; H^k(\mathcal{C}'_\delta))} \right), \end{aligned} \quad (5.48)$$

where $\mathcal{C}'_\delta$ is a fixed larger collar. The commutators with the localisation have order at most three. Formula

$$\mathfrak{D}_\mu \{ \chi(A\zeta)v \} = \chi(A\zeta) \mathfrak{D}_\mu v$$

shows that no unbounded transport commutator is hidden here. Thus the part of the residual-cutoff transition lying outside the common-height collar is an outer-to-inner term only. Scaling $z = A\zeta$ and using H^{k+4} outer maximal regularity gives the weighted Hilbert bound $O(\varepsilon^{1/2})$. For the summable conical component, its n th characteristic block is bounded by

$$C_\delta A_n^{3/2} \left\{ \|\partial_{t_*} w\|_{L^2(J_n; H^k)} + \|w\|_{L^2(J_n; H^{k+4})} \right\}, \quad J_n = [t_*(n), t_*(n+1)].$$

Cauchy–Schwarz and $\sum_n A_n^3 < \infty$ give an $O_\delta(A_0^{3/2})\|w\|_\mathbb{O}$ bound. Thus the combined norm is $O(\varepsilon^{1/2})$ from $\mathbb{O}$ to $\mathbb{Y}_{i,\eta}$; the fourth derivative is included in E_1 precisely for these estimates. Thus the inner, collar and outer estimates patch to an a priori bound of the form (5.47), up to a compact lower-order term.

For completeness, uniform removal of that compact term follows by contradiction. If no ε_0 and uniform constant existed, one could take $\varepsilon_j \downarrow 0$ and a normalised sequence whose image tends to zero. Parabolic compactness and a partition adapted to the local scale leave four possible concentration regions. On bounded similarity sets it converges to a homogeneous solution of $\mathcal{P}_i v = 0$ with zero free trace; the controlled future formula then makes the full trace zero, so the solution vanishes.

There is also a possible intermediate end regime $|\zeta_j| \rightarrow \infty$ with $A_j|\zeta_j| \rightarrow 0$. It is excluded as follows. Choose physical radii r_j with $A_j(s_j) \ll r_j \ll \delta$ around a putative concentration and localise by wide logarithmic cutoffs of the form $\chi(A(s)\zeta/r_j)$. Since $A_s = -\mu A$,

$$\mathfrak{D}_\mu \{ \chi(A\zeta/r_j)v \} = \chi(A\zeta/r_j) \mathfrak{D}_\mu v;$$

there is no time–transport commutator. The spatial commutators have order at most three and tend to zero as the logarithmic width tends to infinity. The end version of (4.27), followed by (4.42), bounds the localised graph norm by its initial conical trace, its localised source and the discounted Hilbert norm. All three tend to zero for the bad sequence. The strict inequality $\eta < \mu/2$ is exactly the positive gap to the transported essential end (A.12); hence no weighted homogeneous slope packet remains. This contradicts concentration in the intermediate regime.

On compact outer sets the sequence converges to a homogeneous cap solution with zero initial trace, hence vanishes. Any remaining mass lies in the fixed physical collar, where (5.48) and the two already vanishing neighbouring pieces force it to zero. This contradicts normalisation.

Local inner and outer inverses, first corestricted to one compatible global height and then restricted back to the two charts, give a parametrix on the fibre product. This is not a direct sum of independent local solutions. Its errors are precisely the compact third-order commutators just estimated. A homotopy through the compatible local principal symbols to the inner initial–future operator and the outer initial-value operator shows that the Fredholm index is zero. The a priori estimate gives injectivity and closed range, hence surjectivity and the asserted bounded inverse. Finally the scalar implicit estimate at the waist supplies the last term in (5.47). $\square$

Lemma 5.6 (Smoothness and quadratic remainder). *On a ball in $\mathfrak{X}$ on which the reconstructed radius is positive and $\|\Theta_\beta(v, w)\|_{L^\infty} < \mu/2$, the section (5.44) is C^2 . If $Z = (v, w)$ and*

$$\mathcal{R}(Z) = \mathcal{S}(Z) - \mathcal{S}(0) - D\mathcal{S}(0)Z,$$

then

$$\|\mathcal{R}(Z) - \mathcal{R}(\tilde{Z})\|_{\mathfrak{Y}} \leq C(\|Z\|_{\mathfrak{X}} + \|\tilde{Z}\|_{\mathfrak{X}})\|Z - \tilde{Z}\|_{\mathfrak{X}}. \quad (5.49)$$

Moreover, for zero free trace data, the residual of the fixed-volume approximate family obeys

$$\|\mathcal{S}(0)\|_{\mathfrak{Y}} \leq C \left\{ (A_0/\delta)^{17/2} + (A_0/\delta)^4 \right\}. \quad (5.50)$$

The constants are independent of A_0/δ .

Proof. In rotational graph coordinates the surface diffusion operator is the rational fourth-order expression (2.11); on the cap it is the usual smooth normal-graph operator. Positivity of the radius and the C^1 smallness built into $\mathfrak{X}$ keep every denominator uniformly separated from zero. The one-dimensional Moser estimates, the algebra property of H^{k-2} for $k \geq 10$, and ordinary product estimates on M_o show that the quasilinear coefficients are C^2 from the section ball to the forcing spaces. The load-bearing Hilbert estimate is, for a typical principal coefficient a ,

$$\begin{aligned} & \|\{a(U' + v') - a(U' + \tilde{v}')\}\partial_\zeta^4 v\|_{L_\eta^2 H_2^{k-2}} \\ & \leq C\|v - \tilde{v}\|_{L_\eta^\infty H_2^k}\|v\|_{L_\eta^2 H_2^{k+2}}. \end{aligned}$$

The second factor supplies one additional $e^{-\eta s}$, so the product lies in the η -weighted source space. Polarisation treats differences of two remainders. On the moving cylinder the same product calculation is built into the graph/source pair $\mathcal{G}_R^k \rightarrow \mathcal{H}_R^k$: a source derivative of order j uses a solution derivative of order $j + 4$ with exactly the same weight in (4.34). Thus there is no derivative loss and no endpoint smoothing assertion. On the discounted inner part the n th source block is $O(e^{-(2\eta-4a)n})$; on the conical end the coefficient and curvature remainders are $O(e^{-4an})$. Both sequences are summable, so the nonlinear remainder takes values in $\mathbb{H}_{R,1}^k$ and obeys the same quadratic difference estimate there.

The factorisation through $\mathfrak{R}_A[u]^{-1}$ above is not used here, because its composition operator would lose a derivative at equal regularity. Instead write the fixed-coordinate expression $\mathfrak{R}_A[0]\{\beta\mathcal{T}_A^\sharp(u) - \mathcal{S}_A^\sharp(u)\}$ directly in the base normal coordinate. It is a quasilinear fourth-order differential expression plus the material derivative. The coefficient of $\mathcal{P}_i v$ has the form $I + b_A(u, \partial u)$ with $b_A(0, 0) = 0$. The scaled Moser estimate and the conical graph-domain term in (5.36) give, for $Z = (v, w)$ and $\tilde{Z} = (\tilde{v}, \tilde{w})$,

$$\begin{aligned} & \|b_A(u_Z, \partial u_Z)\mathcal{P}_i v - b_A(u_{\tilde{Z}}, \partial u_{\tilde{Z}})\mathcal{P}_i \tilde{v}\|_{\mathbb{H}_{R,1}^k} \\ & \leq C(\|Z\|_{\mathfrak{X}} + \|\tilde{Z}\|_{\mathfrak{X}})\|Z - \tilde{Z}\|_{\mathfrak{X}}. \end{aligned}$$

All remaining terms use the four-spatial-derivative gap between $\mathcal{G}_R^k$ and $\mathcal{H}_R^k$. The exact chart $q_A = \mathfrak{N}_A^{-1}(u_A)$ is used only for conjugacy and geometry, not as an equal-regularity source composition. Thus the fixed-coordinate section is C^2 into the summable conical target without derivative loss.

The speed map is C^2 by (5.29), and the cutoff and extension maps are fixed. Finally Lemma 4.5 and $\sigma_N < 2\eta - 4a$ control the future Duhamel integral in the full weighted space. This proves (5.49), including the waist and clock terms.

The inner matching defect is bounded by (5.12); with $m = 2$ this gives the first term in (5.50). Its differentiated annular bounds (5.9) decay geometrically from one characteristic block to the next and give the same estimate in $\mathbb{H}_{R,1}^k$. On the cap, (5.9) gives a physical source of size $O(A^4)$. Since $dt_* = A^4 ds$,

$$\|\mathcal{Q}_A\|_{\mathbb{V}_o}^2 \leq C_\delta \int_0^\infty A(s)^{12} ds \leq C_\delta A_0^{12}.$$

This is $O_\delta(\varepsilon^6)$ and hence is bounded by the coarser second term in (5.50) for $0 < \varepsilon \leq 1$. $\square$

Theorem 5.7 (Fixed-interface half-line section). *Take $k \geq 10$, $m = 2$, and choose the weights as in (5.35). Impose the waist gauge $v(0, s) = 0$, use the compatible global-height fibre product, prescribe the free compatible trace at $A = A_0$, and impose the controlled future condition through weighted decay. Then, for A_0/δ sufficiently small, the linearised section is an isomorphism from $\mathfrak{X}$ to $\mathfrak{Y}$ and*

$$\|L^{-1}Y\|_{\mathfrak{X}} + \|D\Theta_\beta(0)L^{-1}Y\|_{\mathfrak{Z}_\beta} \leq C\|Y\|_{\mathfrak{Y}}, \quad L = D\mathcal{S}(0), \quad (5.51)$$

with C independent of A_0 . With zero free trace, the nonlinear section has a unique zero in the ball of radius

$$C\{(A_0/\delta)^{17/2} + (A_0/\delta)^4\}.$$

After reducing A_0/δ , this selected ball also satisfies $\|\Theta_\beta(v, w)\|_{L^\infty} < \mu/2$; hence the actual clock in (5.15) is positive and uniformly comparable with the base clock. For a prescribed free trace b , add $C\|b\|_{\mathcal{T}^\#}$ to this radius. For the homogeneous selected data used below, the resulting normal displacement is $o(1)$ relative to the local geometric scale uniformly as $A_0 \downarrow 0$.

Proof. The linear estimate and surjectivity are exactly Lemma 5.5. Let $\mathcal{R}(Z) = \mathcal{S}(Z) - \mathcal{S}(0) - LZ$. For prescribed traces we use $\mathcal{S}_b$; its derivative is still L . Zeros are fixed points of

$$\mathcal{T}_b(Z) := -L^{-1}\{\mathcal{S}_b(0) + \mathcal{R}(Z)\}. \quad (5.52)$$

By Lemma 5.6, this map has Lipschitz constant at most $C\rho$ on the radius- ρ ball. Moreover

$$\|\mathcal{S}_b(0)\|_{\mathfrak{Y}} \leq C\{\varepsilon^{17/2} + \varepsilon^4 + \|b\|_{\mathcal{T}^\#}\}.$$

Choose ρ to be twice the right side multiplied by the inverse bound, and then reduce ε and the trace data so that $C\rho < 1/2$. The contraction theorem gives the unique zero and all asserted estimates. The scalar implicit estimate for (5.29) gives the $\mathfrak{Z}_\beta$ bound and, after one further reduction, positivity of the clock. Finally, weighted membership and (5.34) imply $\mathcal{B}_c = 0$ at the zero. $\square$

Lemma 5.8 (Joint future–volume selection). *Let $\psi_1, \dots, \psi_N$ be the compactly supported functions from Lemma 4.7, and let ϕ be the outer field from Lemma 5.3. Extend ϕ to the doubled cap so that its support is disjoint from the matching collar.*

For each sufficiently small free trace $b \in \mathcal{T}_f^\#$, denote the zero of the full section supplied by Theorem 5.7 by

$$Z_\varepsilon[b] = (v_\varepsilon[b], w_\varepsilon[b])$$

and set

$$\Phi_\varepsilon^{\text{full}}(b) := P_c v_\varepsilon[b](\cdot, 0). \quad (5.53)$$

This is a C^1 map. Put $p_\phi = (0, \phi) \in \mathcal{T}_f^\#$ and choose a closed complement

$$\mathcal{T}_f^\# = \mathcal{H} \oplus \text{span}\{p_\phi\}.$$

For $h = (h_i, h_o) \in \mathcal{H}$, put

$$q(a) = \sum_{j=1}^N a_j \psi_j,$$

$$\tau(h, a, \lambda) = h + (q(a), \lambda\phi) \in \mathcal{T}^\#,$$

$$b(h, a, \lambda) = \mathcal{P}_f \tau(h, a, \lambda) = h + \lambda p_\phi.$$

The support choices make $\tau(h, a, \lambda)$ a compatible full trace. Define

$$\mathcal{G}_c^\varepsilon(h, a, \lambda) := P_c q(a) - \Phi_\varepsilon^{\text{full}}(b(h, a, \lambda)) \in E_c, \quad (5.54)$$

$$u_0^\varepsilon(h, a, \lambda) := h_{A_0}[h_i + q(a), h_o + \lambda\phi], \quad (5.55)$$

$$\mathcal{G}_V^\varepsilon(h, a, \lambda) := \text{Vol}(\overline{X}_{A_0} + u_0^\varepsilon(h, a, \lambda)\overline{\nu}_{A_0}) - V_0. \quad (5.56)$$

For $\varepsilon = A_0/\delta$ sufficiently small, the $N + 1$ equations

$$\mathcal{G}_c^\varepsilon(h, a, \lambda) = 0, \quad \mathcal{G}_V^\varepsilon(h, a, \lambda) = 0 \quad (5.57)$$

have, for every sufficiently small $h \in \mathcal{H}$, a unique small solution $(a_\varepsilon(h), \lambda_\varepsilon(h))$, depending C^1 on h . At this solution the corresponding exact invariant family has enclosed volume V_0 for every $0 < A \leq A_0$. The resulting traces form a C^1 submanifold of codimension $N + 1$ in $\mathcal{T}^\#$ and of codimension N in its fixed-volume hypersurface.

Proof. The inverse and quadratic estimates in [Theorem 5.7](#) and [Lemma 5.6](#) give the C^1 solution map and hence (5.53). If $Z_\varepsilon[b(h, a, \lambda)] = (v, w)$, the future Duhamel identity gives

$$\mathcal{G}_c^\varepsilon(h, a, \lambda) = P_c q(a) + \int_0^\infty e^{-\tau \mathcal{L}_c} P_c F_i(v, w)(\tau) \, d\tau.$$

Thus $\mathcal{G}_c^\varepsilon = 0$ is precisely the system of N controlled future conditions. At a zero, the actual trace has free part $b(h, a, \lambda)$ and controlled part $P_c q(a)$; by (5.43) it is therefore exactly $\tau(h, a, \lambda)$.

The map

$$B : \mathbb{R}^N \longrightarrow E_c, \quad Ba = \sum_{j=1}^N a_j P_c \psi_j,$$

is an isomorphism by [Lemma 4.7](#). By the definition of the lift, $\mathcal{P}_f(\psi_j, 0) = 0$. Hence the free datum is independent of a , while the outer datum supported away from the collar has an $o(1)$ effect on the controlled trace. The local parametrices converge in operator norm on this one-dimensional outer span, and therefore

$$D_a \mathcal{G}_c^\varepsilon(0, 0, 0) = B, \quad D_\lambda \mathcal{G}_c^\varepsilon(0, 0, 0) = o(1).$$

The first variation of enclosed volume and the choice of ϕ give

$$D_\lambda \mathcal{G}_V^\varepsilon(0, 0, 0) = \int_{\Sigma_0} \phi \, d\mu_{\Sigma_0} + o(1) =: c_\phi + o(1), \quad c_\phi \neq 0.$$

Each ψ_j has fixed compact support in the similarity variable. Its physical normal height is $O(A_0)$ on an area $O(A_0^2)$, so $D_a \mathcal{G}_V^\varepsilon(0, 0, 0) = O(A_0^3) = o(1)$. Hence

$$D_{(a, \lambda)} \begin{pmatrix} \mathcal{G}_c^\varepsilon \\ \mathcal{G}_V^\varepsilon \end{pmatrix} (0, 0, 0) = \begin{pmatrix} B + o(1) & o(1) \\ o(1) & c_\phi + o(1) \end{pmatrix}, \quad (5.58)$$

which is invertible for small ε . At $h = 0$, the value of the left-hand side of (5.57) at $(a, \lambda) = (0, 0)$ tends to zero with the matching defect. The parameter-dependent finite-dimensional implicit function theorem yields unique small C^1 functions $(a_\varepsilon(h), \lambda_\varepsilon(h))$. Projection onto the direct summand $\mathcal{H}$ recovers h from the resulting full trace, so this parametrisation is a C^1 embedding after shrinking the ball. Since

$$\mathcal{T}^\# = \mathcal{H} \oplus \text{span}\{p_\phi\} \oplus \mathcal{J}_c E_c,$$

its image has codimension $N + 1$ in $\mathcal{T}^\#$. The volume equation is transverse to p_ϕ , so the same image has codimension N within the fixed-volume hypersurface. The volume chart is used here only for this transverse initial direction; no slice-by-slice correction is introduced.

Let Ω_A be the region bounded by the selected invariant surface. The invariance equation and the first variation of volume give

$$\beta(A) \frac{d}{dA} \text{Vol}(\Omega_A) = \int_{\partial\Omega_A} \beta(A) \langle \partial_A X_A, \nu_A \rangle \, d\mu_A \quad (5.59)$$

$$= \int_{\partial\Omega_A} \mathcal{S}_A^\# \, d\mu_A = - \int_{\partial\Omega_A} \Delta H \, d\mu_A = 0. \quad (5.60)$$

Since $\beta(A) < 0$, the volume is constant in A and equals V_0 by (5.56). $\square$

Lemma 5.9 (Invariant families generate flows). *Let X_A be a C^1 family of smooth embedded surfaces for $0 < A \leq A_0$ and suppose*

$$\mathcal{S}_A^\# = \beta(A) \langle \partial_A X_A, \nu_A \rangle, \quad \beta(A) < 0. \quad (5.61)$$

If $A_t = \beta(A)$, then, after a time-dependent reparametrisation of the domain, $X_{A(t)}$ is a surface diffusion flow.

Proof. The normal component of $\partial_t X_{A(t)}$ is $\beta(A) \langle \partial_A X_A, \nu_A \rangle$, which equals the surface diffusion normal speed by (5.61). Their difference is tangential. Integrating the negative of this tangential field on the compact parameter manifold produces the required family of diffeomorphisms. $\square$

Lemma 5.10 (Quantitative embeddedness). *There is $\varepsilon_* > 0$, independent of sufficiently small A_0 , such that every corrected meridian produced by the section is a simple loop in $\{r > 0\}$ whenever*

$$\sup_s \mathbf{c}_{1,R}(v; s) + \sup_{t_* \in I_*} \|w(t_*)\|_{C^1(M_o)} < \varepsilon_*. \quad (5.62)$$

Consequently its surface of revolution is an embedded torus.

Proof. Introduce the local length

$$\ell_A(z) = A + |z| \quad (|z| \leq 2\delta), \quad \ell_A = \delta \quad \text{on the outer cap.}$$

The approximate meridians possess a normal tube of radius $c\ell_A$. Indeed, take a hypothetical sequence of first failures, translate to the bad points and dilate by ℓ_A^{-1} . The differentiated profile expansion and fixed-interface estimates give uniform C^2 bounds on every fixed rescaled ball. A subsequence therefore converges to exactly one of three limits, according as $|z|/A$ stays bounded, tends to infinity while $|z|/\delta \rightarrow 0$, or stays in the outer region: a piece of the positive rounded profile U , a straight conical ray, or a piece of the fixed outer cap. The first and third have positive reach on compact subarcs and the second is a line, contradicting the normalised first failure. Compactness of the three limiting regimes makes the resulting $c > 0$ uniform.

On $|z| \leq \delta/2$, use the exact radial coordinate supplied by Lemma 5.4. Since $\bar{r}_A(z) = AU(z/A)$ there, the corrected radial graph is

$$r_{\text{corr}}(z, A) = AU(z/A) + q_A(z), \quad q_A = \mathfrak{N}_A^{-1}\{M_A(Av)\}.$$

The uniform inverse-chart estimate gives $q_A = Av + O(|Av|^2/\ell_A)$ and its corresponding scaled first-derivative bound. Positivity of the profile and (5.62) give $r_{\text{corr}} \geq AU - |q_A| \geq c(A + |z|) > 0$. Moreover

$$|Av(z/A, s)| + \ell_A(z) |\partial_z \{Av(z/A, s)\}| \leq C \mathbf{c}_{1,R}(v; s) \ell_A(z).$$

After decreasing ε_* , the same inequality, with a larger constant, holds with Av replaced by q_A . The radial-normal reparametrisation has uniformly bounded scaled first derivative by Lemma 5.4, so the same estimate measures the geometric C^1 displacement in the normal tube. Thus the correction stays inside this variable-radius tube. On $\delta/2 \leq |z| \leq 2\delta$, the fixed-interface estimates give a C^1 perturbation of a fixed simple conical annulus. On the remaining cap, compactness supplies a positive tubular radius and the outer C^1 norm preserves simplicity. The three conclusions agree on their overlaps because the correction is reconstructed by the fixed partition of unity. Distinct core and cap portions remain separated by a fixed physical distance outside the adjacent overlaps. Thus no new intersection is possible. Since the whole loop lies in $r > 0$, revolution about the axis is an embedding of $\mathbb{T}^2$. $\square$

Theorem 5.11 (Compactification of a positive conical profile). *Suppose that there is a smooth positive even conical profile U satisfying (2.16) and the differentiated tail*

$$U(\zeta) = \alpha\zeta + b\zeta^{-3} + c\zeta^{-7} + O(\zeta^{-11}) \quad (\zeta \rightarrow +\infty).$$

Fix an (α, δ) -admissible terminal meridian Γ_0 , let $N = \dim E_c$, and take $k \geq 10$. There is $A_* > 0$ such that, whenever $0 < A_0 < A_*$ and (5.19) holds, the compatible trace space at scale A_0 contains an infinite-dimensional C^1 submanifold $\mathcal{M}_{\text{pin}}^k$ of codimension $N + 1$. It lies in the fixed-volume hypersurface and has codimension N there. Every smooth trace in this submanifold gives a smooth closed embedded torus of revolution whose surface diffusion flow remains embedded for $t < T$ and develops an axial curvature singularity at a finite time T . Its reflected waist satisfies

$$A(t) = \{4\mu(T - t)\}^{1/4}(1 + o(1)), \quad (5.63)$$

$$\frac{r(A(t)\zeta, t)}{A(t)} \longrightarrow U(\zeta) \quad \text{locally smoothly.} \quad (5.64)$$

The limiting outer cap is a small smooth perturbation of Γ_0 . The submanifold and this limiting cap depend C^1 on small admissible perturbations of Γ_0 . No assumption that U is spectrally stable is required. The N controlled core coordinates and one transverse volume coordinate are determined as C^1 functions of the arbitrary infinite-dimensional free trace.

Proof. Fix Γ_0 and reduce A_0 so that both the matching smallness conditions and (5.19) hold. Use Proposition 5.1 to construct a fixed-volume approximate family $\bar{\Sigma}_A$. The Fredholm splitting in Proposition 4.2, the core evolution and tame estimate in Proposition 4.4, the mixed inverse (5.51), and the quadratic normal-graph expansion give a contraction for (5.13), uniformly for small free traces $h \in \mathcal{H}$. The finite-dimensional compatibility conditions and the initial volume equation are solved jointly, as C^1 functions of h , by Lemma 5.8. Volume then propagates by (5.60). This yields an exact invariant curve with

$$\beta(A) = -\mu A^{-3} + A^{-3}\Theta_\beta(v, w)(s), \quad |\Theta_\beta(v, w)(s)| \leq Ce^{-\eta s}.$$

By Lemma 5.9 this curve, after a tangential reparametrisation, is an exact surface diffusion flow. Its physical clock satisfies

$$T - t(s) = \int_s^\infty \frac{A_0^4 e^{-4\mu\sigma}}{1 - \Theta_\beta(v, w)(\sigma)/\mu} d\sigma = \frac{A(s)^4}{4\mu} \{1 + O(e^{-\eta s})\}.$$

The direct-sum description and codimension assertions follow from Lemma 5.8; since $\mathcal{H}$ is infinite dimensional, so is $\mathcal{M}_{\text{pin}}^k$. The section estimates hold at every integer level $k \geq 10$. For a smooth free trace, solutions constructed at two levels agree in the lower space by uniqueness of the contraction, so the selected invariant curve belongs to the inverse limit of all weighted Sobolev and conical graph spaces. Fourth-order parabolic bootstrapping therefore makes every positive-time slice smooth; choose one such slice as the initial torus. The all-orders mixed estimate gives C_{loc}^∞ convergence to U and smooth outer convergence on compact sets away from the terminal point. Sobolev embedding on the doubled cap and the conical graph bound give (5.62); hence Lemma 5.10 shows that the corrected meridian is a simple loop in $\{r > 0\}$ for every $A > 0$. Finally the azimuthal curvature of the waist is $A^{-1} \rightarrow \infty$, and (5.63) follows from the displayed clock asymptotic. The outer maximal-regularity trace has a terminal value, so the limiting outer cap is a small perturbation of Γ_0 . All matching operators, parametrices, contraction maps and finite-dimensional equations depend C^1 on an admissible cap parameter with the uniform bounds from Lemma 5.2. The parameter-dependent contraction and implicit function theorems give the asserted C^1 cap dependence. $\square$

Thus the conical route has one finite-dimensional input: a rigorous existence and positivity certificate for the classical fundamental profile, reduced to the explicit interval assertions in Theorem 3.9. Neither a numerical eigenvalue count of this profile nor capture from a circular torus lies on the critical path.

Proof of Theorem 1.2. Apply Theorem 5.11 to the profile supplied by Theorem 3.9 and to an arbitrary admissible reference cap Γ_0 . The parameter enclosures are those of the profile theorem. For A_0 below the cap-dependent threshold, including (5.19), the compactification theorem and Lemma 5.8 give the asserted infinite-dimensional family and its codimensions. Each member is an exact invariant curve for surface diffusion, remains embedded for every positive waist radius, and satisfies (5.63)–(5.64). At the waist $r_z = 0$, so the azimuthal principal curvature has magnitude $A(t)^{-1}$. Thus curvature diverges as $t \uparrow T$. By uniqueness of the local smooth semiflow, no smooth continuation through T is possible, and T is the maximal forward time. Smooth convergence on compact subsets separated from the terminal conical point, and C^1 dependence on the reference cap, follow from the outer maximal-regularity construction and parabolic bootstrapping. $\square$

A The terminal Fredholm theorem

We prove the operator assertions used in Proposition 4.2. Throughout this appendix $m = 2$ and $k \geq 2$. It is convenient to begin before imposing the waist normalisation. Put

$$\begin{aligned}\tilde{X}^k &= H_2^k(\mathbb{R})_{\text{even}}, \\ \tilde{D}^k &= \{v \in H_2^{k+4}(\mathbb{R}) : \zeta v' \in H_2^k, v \text{ even}\},\end{aligned}$$

with the graph norm

$$\|v\|_{\tilde{D}^k} = \|v\|_{H_2^{k+4}} + \|\zeta v'\|_{H_2^k}.$$

Let $\mathcal{L}_0 = \mathcal{A}_U + \mu(1 - \zeta \partial_\zeta)$ be the unnormalised operator in (4.5).

Coefficient structure and the end problem

Direct differentiation of (4.3) writes

$$\mathcal{A}_U v = -a_4 v^{(4)} + a_3 v''' + a_2 v'' + a_1 v' + a_0 v, \quad (\text{A.1})$$

where

$$a_4 = \frac{1}{2(1 + U^2)^2}, \quad 0 < a_* \leq a_4 \leq a^* < \infty.$$

The differentiated conical expansion gives, with $q_\alpha = 1 + \alpha^2$,

$$\begin{aligned}a_4 &= \frac{1}{2q_\alpha^2} + O(|\zeta|^{-4}), & a_3 &= O(|\zeta|^{-1}), & a_2 &= O(|\zeta|^{-2}), \\ a_1 &= O(|\zeta|^{-3}), & a_0 &= O(|\zeta|^{-4}),\end{aligned} \quad (\text{A.2})$$

with the corresponding differentiated estimates. Notice that the first error in a_4 still multiplies $v^{(4)}$; it is not compact as a map $\tilde{D}^k \rightarrow \tilde{X}^k$.

For fixed λ , the homogeneous equation $(\mathcal{L}_0 - \lambda)v = 0$ has on either end the algebraic solution and three WKB solutions

$$v_a(\zeta) = |\zeta|^{1-\lambda/\mu} \{1 + O(|\zeta|^{-4})\}, \quad (\text{A.3})$$

$$v_j(\zeta) = |\zeta|^{\lambda/(3\mu)-5/3} \exp\{\varrho_j \kappa |\zeta|^{4/3}\} \{1 + O(|\zeta|^{-4/3})\}, \quad \varrho_j^3 = -1. \quad (\text{A.4})$$

Exactly one WKB branch decays and two grow. Moreover, $v_a \in H_2^k$ precisely when

$$\Re \lambda > -\frac{\mu}{2}, \quad (\text{A.5})$$

because its weighted square is asymptotic to $|\zeta|^{-2-2\Re \lambda/\mu}$. The asymptotic integration and dichotomy framework used below goes back to Levinson and Palmer [29, 38]; the coefficients and weighted threshold here are those of the present conical operator.

Lemma A.1 (Uniform end dichotomy). *Let K be a compact subset of $\{\Re \lambda > -\mu/2\}$. For R sufficiently large, the first-order system associated with $(\lambda - \mathcal{L}_0)v = f$ on either end has a two-dimensional admissible bundle, spanned asymptotically by (A.3) and the decaying WKB branch, and a two-dimensional inadmissible bundle. Its one-sided Green operator is bounded from the weighted data space to the weighted graph space uniformly for $\lambda \in K$, and depends analytically on λ .*

Proof. Put $a_\infty = (2(1 + \alpha^2)^2)^{-1}$ and $W = (v, v', v'', v''')^T$. The exact first-order system is

$$W' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ (a_0 + \mu - \lambda)/a_4 & (a_1 - \mu\zeta)/a_4 & a_2/a_4 & a_3/a_4 \end{pmatrix} W + a_4^{-1}(0, 0, 0, f)^T. \quad (\text{A.6})$$

Let $c_\infty = (\mu/a_\infty)^{1/3}$ and let $\varrho_j^3 = -1$. An explicit leading conjugating matrix is formed by the algebraic column

$$t_a(\zeta) = \left(1, \frac{1 - \lambda/\mu}{\zeta}, \frac{(1 - \lambda/\mu)(-\lambda/\mu)}{\zeta^2}, \frac{(1 - \lambda/\mu)(-\lambda/\mu)(-1 - \lambda/\mu)}{\zeta^3}\right)^T$$

and the three WKB columns

$$t_j(\zeta) = (1, p_j, p_j^2, p_j^3)^T, \quad p_j = \varrho_j c_\infty \zeta^{1/3}.$$

Its determinant is separated from zero after the natural diagonal scaling, uniformly for $\lambda \in K$ and large ζ . Substitution in (A.6), followed by one algebraic near-identity correction, block-diagonalises the system into the algebraic equation below and

$$Z_x = \kappa \text{diag}(-1, e^{i\pi/3}, e^{-i\pi/3})Z + x^{-1}D(\lambda)Z + R(x, \lambda)Z + F, \quad \|R(x, \lambda)\| \leq Cx^{-1-\epsilon}, \quad (\text{A.7})$$

where $x = \zeta^{4/3}$ and $\kappa = 3c_\infty/4$. The coefficient decay (A.2) gives the uniform remainder bound, also after the differentiations required by the graph norm.

For the algebraic block use $y = \log |\zeta|$ and $w = e^{-3y/2}v(e^y)$. The limiting equation is

$$\{\lambda - \mu(-\tfrac{1}{2} - \partial_y)\}w = g,$$

whose forward kernel is

$$K_a(y, \eta) = \mu^{-1} \mathbf{1}_{\eta \leq y} e^{-(\lambda + \mu/2)(y - \eta)/\mu}$$

and has norm at most $\text{dist}(K, \{\Re \lambda \leq -\mu/2\})^{-1}$. For (A.7), factor the explicit diagonal power $x^{D(\lambda)}$ and use

$$\begin{aligned} K_-(x, s) &= \mathbf{1}_{s \leq x} e^{-\kappa(x-s)}, \\ K_\pm(x, s) &= -\mathbf{1}_{s \geq x} e^{\kappa e^{\pm i\pi/3}(x-s)}. \end{aligned}$$

Their L^1 operator norms are bounded by the reciprocal real-part gaps. The $Cx^{-1-\epsilon}$ remainder is a contraction after increasing the end base. Differentiation of these Volterra equations gives the weighted graph estimate; analytic dependence follows from uniform convergence of the Neumann series. The algebraic and decaying WKB columns are admissible and the other two are excluded, proving the asserted dichotomy and Green map. $\square$

Lemma A.2 (Fredholm estimate). *For every compact $K \Subset \{\Re \lambda > -\mu/2\}$ there are $C, R < \infty$ such that*

$$\|v\|_{\tilde{D}^k} \leq C \left\{ \|(\lambda - \mathcal{L}_0)v\|_{\tilde{X}^k} + \|v\|_{H^{k+3}(-R, R)} \right\} \quad (\text{A.8})$$

for $\lambda \in K$ and $v \in \tilde{D}^k$. Consequently $\lambda - \mathcal{L}_0$ is Fredholm of index zero throughout that half-plane.

Proof. Apply Lemma A.1 on the two ends and the ordinary fourth-order estimate on a compact core. Fixed cutoffs produce only third-order commutators supported in a compact annulus, giving (A.8). The embedding $\tilde{D}^k \hookrightarrow H^{k+3}(-R, R)$ is compact. The same two one-sided Green operators and a compact-core inverse form a two-sided parametrix; their cutoff commutators are compact by the displayed embedding. Thus the kernel and cokernel are finite and the range is closed.

The index follows from the end count. On the even half-line there are two admissible modes at infinity, while regular evenness imposes $v'(0) = v'''(0) = 0$; hence the matching problem is square. Equivalently this is the exponential-dichotomy index formula [38]. The index is therefore zero wherever the pencil is Fredholm. $\square$

Generation and the vertical resolvent

Lemma A.3 (Maximal realisation). *The closure of $\mathcal{L}_0$ on $C_c^\infty(\mathbb{R})_{\text{even}}$ has domain $\tilde{D}^k$ and generates a C_0 -semigroup on $\tilde{X}^k$.*

Proof. We first identify the core. Let $\chi \in C_c^\infty(\mathbb{R})$ be even, equal to one on $[-1, 1]$, and put

$$\chi_R(\zeta) = \chi\left(\frac{\log\langle\zeta\rangle}{\log R}\right), \quad R > e.$$

Then $\chi_R v \rightarrow v$ in H_2^{k+4} by dominated convergence. Moreover $|\zeta \partial_\zeta \chi_R| \leq C/\log R$, with the analogous bounds after commuting k derivatives, and hence

$$\|\zeta \partial_\zeta \{(1 - \chi_R)v\}\|_{H_2^k} \leq \|(1 - \chi_R)\zeta v'\|_{H_2^k} + \frac{C}{\log R} \|v\|_{H_2^{k+4}} \rightarrow 0.$$

Even mollification of $\chi_R v$ proves that $C_c^\infty(\mathbb{R})_{\text{even}}$ is a core for $\tilde{D}^k$.

Twice integrating the principal term in (A.1) by parts, then commuting through the first k derivatives, gives the weighted Gårding estimate

$$\Re \langle \mathcal{L}_0 v, v \rangle_{\tilde{X}^k} \leq -c \|v\|_{H_2^{k+2}}^2 + C \|v\|_{\tilde{X}^k}^2. \quad (\text{A.9})$$

All logarithmic derivatives of the weight are bounded. The drift causes no loss, since

$$\Re \int -\mu \zeta v' \bar{v} \langle \zeta \rangle^{-4} d\zeta = \frac{\mu}{2} \int (\zeta \langle \zeta \rangle^{-4})' |v|^2 d\zeta$$

and $[\partial_\zeta^j, \zeta \partial_\zeta] = j \partial_\zeta^j$. Local elliptic regularity and the two end estimates identify the closure with the maximal graph domain. Estimate (A.9) gives injectivity for sufficiently large positive λ ; Lemma A.2 and index zero give surjectivity. After a scalar shift, Lumer–Phillips [31] proves generation. $\square$

Lemma A.4 (Uniform vertical resolvent). *For every $\delta > 0$ there are $C_\delta, R_\delta < \infty$ such that*

$$\|v\|_{\tilde{X}^k} \leq C_\delta \|(\lambda - \mathcal{L}_0)v\|_{\tilde{X}^k} \quad (\text{A.10})$$

whenever $\Re \lambda \geq -\mu/2 + \delta$ and $|\Im \lambda| \geq R_\delta$.

Proof. On the logarithmic ends, the real part of $\lambda - \mu(-\frac{1}{2} - \partial_y)$ has gap δ . The fourth-order term has the dissipative sign, and the errors in (A.2) are absorbed by taking the end cutoff large. On a fixed core the symbol is

$$\lambda + a_4(\zeta)\xi^4 + i\mu\zeta\xi.$$

For a cutoff supported in $|\zeta| < 2R$, freezing a_4 on finitely many coordinate intervals and using the constant-coefficient Fourier multiplier estimate gives

$$\begin{aligned} & |\lambda| \|\chi v\|_{H^k} + |\lambda|^{1/2} \|\chi v\|_{H^{k+2}} + \|\chi v\|_{H^{k+4}} \\ & \leq C \left(\|\chi(\lambda - \mathcal{L}_0)v\|_{H^k} + \|v\|_{H^{k+3}(R < |\zeta| < 2R)} \right). \end{aligned} \quad (\text{A.11})$$

The bounded transport coefficient on the core is lower order here. On the ends, conjugation by $v(e^y) = e^{3y/2}w(y)$ makes the transport part $-\mu/2 + \mu\partial_y$; Fourier transformation in y gives the gap δ , while the fourth-order part has the dissipative sign. The coefficient errors are absorbed after increasing R .

If the asserted estimate failed, take a normalised bad sequence. A core–end partition and (A.11) show that no fixed proportion of its norm can remain in the core. Translate each remaining end piece in y . Weak compactness and the uniform local graph estimate give a nonzero limit solving the constant transport resolvent equation on $\mathbb{R}$; the Fourier gap forces that limit to vanish, a contradiction. Compactly supported partition commutators are absorbed by the parameter gain. This proves (A.10). $\square$

The exact essential boundary and waist quotient

Fredholmness gives $s_{\text{ess}}(\mathcal{L}_0) \leq -\mu/2$. For the reverse inequality, fix $\tau \in \mathbb{R}$ and choose

$$w_n(y) = L_n^{-1/2} \chi\left(\frac{y - y_n}{L_n}\right) e^{-i\tau y/\mu}, \quad y_n, L_n \rightarrow \infty, \quad L_n e^{-4y_n} \rightarrow 0.$$

Pull back by $v(e^y) = e^{3y/2}w(y)$ and reflect evenly to the negative end. After normalisation in $\tilde{X}^k$ this is a weakly null Weyl sequence and

$$\|\{\mathcal{L}_0 + \mu/2 - i\tau\}v_n\|_{\tilde{X}^k} \rightarrow 0.$$

The fourth-order part and all coefficient errors vanish in the translated logarithmic limit. Hence

$$-\frac{\mu}{2} + i\mathbb{R} \subset \sigma_{\text{ess}}(\mathcal{L}_0), \quad s_{\text{ess}}(\mathcal{L}_0) = -\frac{\mu}{2}. \quad (\text{A.12})$$

It remains to impose the dynamic neck normalisation. Since $k \geq 2$, the trace $\ell(v) = v(0)$ is continuous. With $DU = U - \zeta U'$,

$$\mathcal{L}_0 DU = 4\mu DU, \quad DU(0) = 1,$$

and there are topological decompositions

$$\tilde{X}^k = \mathcal{X}_2^k \oplus \text{span}\{DU\}, \quad \tilde{D}^k = \mathcal{D}_{2,k} \oplus \text{span}\{DU\}.$$

Relative to them,

$$\mathcal{L}_0 = \begin{pmatrix} \mathcal{L} & 0 \\ b & 4\mu \end{pmatrix}, \quad b(v) = (\mathcal{L}_0 v)(0). \quad (\text{A.13})$$

Thus $\mathcal{L}$ generates the quotient semigroup $e^{s\mathcal{L}}v = \Pi e^{s\mathcal{L}_0}v$ and inherits the Fredholm index, essential boundary and high-frequency estimate. This exact triangular identity also shows why the scale vector is removed without discarding a possible additional eigenvector or Jordan chain at the same eigenvalue.

Finally analytic Fredholm and Riesz-projection theory [23] makes the quotient resolvent meromorphic in $\Re \lambda > -\mu/2$. Generation bounds its spectrum on the right, while Lemma A.4 prevents poles from escaping vertically in a closed sub-half-plane. Hence only finitely many eigenvalues lie to the right of $-\omega$. On the complementary Riesz subspace the vertical estimate controls large imaginary part, the pole-free compact rectangle controls bounded imaginary part,

and Hille–Yosida controls large positive real part. The stable resolvent is therefore uniform on $\{\Re \lambda \geq -\eta\}$ for $0 < \eta < \omega$; the Gearhart–Prüss theorem [16, 39] gives

$$\|e^{s\mathcal{L}}P_s\| \leq C_\eta e^{-\eta s}.$$

This completes the proof of Proposition 4.2.

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