

Numerical Supplement for the Positive Conical Similarity Profile

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Abstract

This supplement isolates the computer-assisted part of the construction of a positive conical similarity profile for axisymmetric surface diffusion. It gives the exact fourth-order equation, the four-term asymptotic profile, the real compactified normal form, the interval quantities evaluated by the accompanying Python code, and the fixed-point inequalities that turn those quantities into a tail theorem. Every constant is marked as either: (i) inherited from the manuscript, (ii) freshly chosen in the present proof, or (iii) freshly computed in a particular Arb receipt. For the centre calculation it gives a rigorous shifted Taylor-tail enclosure, an affine relaxation retaining the three parameter noises, and a strict componentwise correction-tube argument. The six directly evaluated signed faces then give the Poincaré–Miranda zero and the matched global positive profile.

1 Scope

The manuscript constructs an even similarity profile U satisfying

$$\mathcal{F}[U] + \mu(U - \zeta U') = 0, \quad U(0) = 1, \quad \mu > 0, \quad (1)$$

where

$$\mathcal{F}[U] = \frac{1}{2U} \frac{d}{d\zeta} \left[\frac{U}{\sqrt{1 + (U')^2}} \frac{d}{d\zeta} \left(\frac{1}{U\sqrt{1 + (U')^2}} - \frac{U''}{(1 + (U')^2)^{3/2}} \right) \right]. \quad (2)$$

The numerical proof joins a regular centre solution on $0 \leq \zeta \leq 20$ to an asymptotically conical tail on $\zeta \geq 20$. This supplement certifies both parts and their matching at $\zeta = 20$.

We use the following terms.

Inherited A formula or decimal printed in the manuscript or its stand-alone rewrite. It is not thereby recomputed by the present code.

Fresh chosen An exact rational proof parameter encoded in the present source.

Fresh computed

An outward-rounded Arb enclosure stored in a particular canonical receipt.

The fresh proof intentionally uses more conservative constants than those quoted in the manuscript. Accordingly, its larger endpoint displacement is used in every centre face calculation below.

2 Exact scalar and flux equations

Put $q = U'$, $\omega = 1 + q^2$, $r = U''$, and $s = U'''$. Solving (1) for the fourth derivative gives the exact rational expression

$$U^{(4)} = 2\omega^2 \left[\left(\frac{5qr}{\omega^3} - \frac{q}{U\omega^2} \right) s + \frac{3(1-5q^2)r^3}{2\omega^4} + \frac{(6q^2-1)r^2}{2U\omega^3} + \frac{(q^2-1)r}{2U^2\omega^2} + \frac{q^2}{2U^3\omega} + \mu(U - \zeta q) \right]. \quad (3)$$

This is the expression differentiated symbolically in `symbolic_tail.py`, included in the accompanying certificate archive. The archive is available at

https://glenw83.github.io/conical_pincher_certificate.zip.

It is also useful to integrate the centre in a nonsingular flux formulation. With

$$S_q = \sqrt{1+q^2}, \quad K = \frac{1}{US_q} - \frac{q'}{S_q^3}, \quad J = \frac{U}{S_q} K', \quad (4)$$

equation (1) is equivalent, while $U > 0$, to

$$U' = q, \quad q' = \frac{1+q^2}{U} - (1+q^2)^{3/2}K, \quad K' = \frac{S_q J}{U}, \quad J' = -2\mu U(U - \zeta q). \quad (5)$$

The shooting data are

$$(U, q, K, J)(0) = (1, 0, 1 - \gamma, 0). \quad (6)$$

Equations (3) and (5) provide a cross-check between the tail and centre implementations.

3 Four-term conical truncation

For $\alpha > 0$, seek a formal tail

$$U(\zeta) \sim \alpha\zeta + \sum_{n \geq 1} c_n \zeta^{1-4n}. \quad (7)$$

If $[\zeta^m]H$ denotes coefficient extraction in a formal Laurent series, then the inherited recursion is

$$c_n = -\frac{1}{4n\mu} [\zeta^{1-4n}] \mathcal{F} \left[\alpha\zeta + \sum_{j=1}^{n-1} c_j \zeta^{1-4j} \right]. \quad (8)$$

Exact expansion of (8) yields

$$c_1 = -\frac{1}{8\mu\alpha(1+\alpha^2)}, \quad (9)$$

$$c_2 = -\frac{21(11\alpha^2+1)}{128\mu^2\alpha^3(1+\alpha^2)^3}, \quad (10)$$

$$c_3 = -\frac{3(99985\alpha^4+10972\alpha^2+183)}{1024\mu^3\alpha^5(1+\alpha^2)^5}, \quad (11)$$

$$c_4 = -\frac{6104617253\alpha^6+715737891\alpha^4+16291251\alpha^2+91661}{32768\mu^4\alpha^7(1+\alpha^2)^7}. \quad (12)$$

The first two closed forms and the recursion are inherited. The last two closed forms are fresh exact consequences checked by the code.

Set

$$U_4(\zeta) = \alpha\zeta + \sum_{n=1}^4 c_n \zeta^{1-4n}. \quad (13)$$

Writing $t = \zeta^{-4}$, the exact checker clears the nonzero denominator in (3), substitutes $U_4 = \zeta u(t)$, and verifies that the residual numerator has zero coefficients in degrees 0, 1, 2, 3. Its degree four coefficient is nonzero. Equivalently, the first possible unbalanced power has the order used in the forcing estimate

$$|g(x)| \leq K_g x^{-\beta-1}, \quad \beta = \frac{57}{4}. \quad (14)$$

No floating-point arithmetic is used in this cancellation.

4 Compactified real normal form

Define

$$x = \zeta^{4/3}, \quad w = x^{-1/4} = \zeta^{-1/3}, \quad x_0 = 20^{4/3}, \quad (15)$$

and

$$\kappa = \frac{3}{4}(2\mu(1 + \alpha^2)^2)^{1/3}. \quad (16)$$

For $V = U - U_4$, let

$$Y = (V, V_x, V_{xx}, V_{xxx})^\top. \quad (17)$$

The complex eigenvalues of the limiting first-order system are

$$0, \quad -\kappa, \quad \kappa e^{i\pi/3}, \quad \kappa e^{-i\pi/3}. \quad (18)$$

The implementation works in real coordinates $Z = (Z_0, Z_s, Z_r, Z_i)^\top$, where the last two coordinates represent the conjugate pair. It uses the transformation

$$Y = A(w, \kappa)Z, \quad (19)$$

with

$$A = \begin{pmatrix} 1 & \frac{\kappa - \frac{5}{2}w^4}{\kappa} & 2 + \frac{5w^4}{2\kappa} & \frac{5\sqrt{3}w^4}{2\kappa} \\ \frac{3}{4}w^4 & -\kappa + \frac{5}{4}w^4 & \kappa + \frac{5}{2}w^4 & -\sqrt{3}\kappa \\ 0 & \kappa^2 & -\kappa^2 & -\sqrt{3}\kappa^2 \\ 0 & \kappa^2(-\kappa - \frac{5}{4}w^4) & \frac{\kappa^2(-8\kappa + 5w^4)}{4} & \frac{5\sqrt{3}\kappa^2 w^4}{4} \end{pmatrix}. \quad (20)$$

Exact symbolic arithmetic proves

$$\det A = 6\sqrt{3}\kappa^6 > 0 \quad (21)$$

throughout the parameter box and verifies the explicit inverse used by the interval program.

In these coordinates the exact equation takes the form

$$Z_x = \left(\Lambda_{\mathbb{R}} + \frac{D}{x} \right) Z + \mathcal{R}(x)Z + \mathcal{Q}(x, Z) + g(x), \quad (22)$$

where

$$\Lambda_{\mathbb{R}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\kappa & 0 & 0 \\ 0 & 0 & \kappa/2 & -\sqrt{3}\kappa/2 \\ 0 & 0 & \sqrt{3}\kappa/2 & \kappa/2 \end{pmatrix}, \quad D = \text{diag} \left(\frac{3}{4}, -\frac{5}{4}, -\frac{5}{4}, -\frac{5}{4} \right). \quad (23)$$

The nonlinear terms are defined by the exact vector field:

$$\begin{aligned}
g(x) &= f_Z[U_4](x, 0), \\
\mathcal{R}_{\text{cone}}(x) &= D_Z f_Z[\alpha\zeta](x, 0) - \left(\Lambda_{\mathbb{R}} + \frac{D}{x} \right), \\
\mathcal{R}(x) &= D_Z f_Z[U_4](x, 0) - \left(\Lambda_{\mathbb{R}} + \frac{D}{x} \right), \\
\mathcal{Q}(x, Z) &= f_Z[U_4](x, Z) - f_Z[U_4](x, 0) - D_Z f_Z[U_4](x, 0)Z.
\end{aligned} \tag{24}$$

Here f_Z is obtained from (3), the chain rule

$$\begin{aligned}
V_\zeta &= aV_x, \\
V_{\zeta\zeta} &= a^2V_{xx} + a'V_x, \\
V_{\zeta\zeta\zeta} &= a^3V_{xxx} + 3aa'V_{xx} + a''V_x, \\
V_{\zeta\zeta\zeta\zeta} &= a^4V_{xxxx} + 6a^2a'V_{xxx} + (3(a')^2 + 4aa'')V_{xx} + a'''V_x,
\end{aligned} \tag{25}$$

where

$$a = \frac{4}{3w}, \quad a' = \frac{4}{9}w^2, \quad a'' = -\frac{8}{27}w^5, \quad a''' = \frac{40}{81}w^8, \tag{26}$$

and the identity

$$f_Z(x, Z) = A^{-1}(f_Y(x, AZ) - A_x Z). \tag{27}$$

Thus every expression enclosed by Arb is an algebraic consequence of the original profile equation.

5 Interval domain and proof quantities

The fresh computation uses

$$\alpha \in [1.0370, 1.0372], \quad \mu \in [0.03028, 0.03031], \quad w \in \left[0, \frac{737}{2000}\right]. \tag{28}$$

It partitions this box into $1024 \times 4 \times 4 = 16384$ rational slabs and evaluates each slab at 256-bit Arb precision. Endpoints are exact rationals; all transcendental and algebraic operations are outward rounded.

The real conjugate-pair norm is

$$|Z|_c^2 = Z_0^2 + Z_s^2 + 2Z_r^2 + 2Z_i^2. \tag{29}$$

For matrices the program uses the Frobenius norm of $W^{1/2}MW^{-1/2}$, $W = \text{diag}(1, 1, 2, 2)$, which bounds the induced norm. The analogous weighted Frobenius norm is used for bilinear maps.

Put $Z = w^{57}E$. On the complete fixed-point tube the fresh chosen component envelope is

$$|E_j| \leq R + \varepsilon_s 20^{19}, \quad R = 10^{14}, \quad \varepsilon_s = 10^{-11}. \tag{30}$$

The regular state variables passed to the Hessian evaluator are u, q, b_2, b_3 , where

$$u = w^3U, \quad q = U', \quad w^2U'' = w^{17}b_2, \quad w^3U''' = w^{21}b_3. \tag{31}$$

These substitutions remove every apparent singularity at $w = 0$.

Definition 5.1 (Fresh tail acceptance tests). A tail receipt is successful if its outward-rounded enclosures prove all of the following strict inequalities:

$$\sup x^2 \|\mathcal{R}_{\text{cone}}\| < 30, \quad \sup x^{9/4} |Z[U_4 - \alpha\zeta]|_c < \frac{7}{2}, \quad (32)$$

$$\sup_{\text{Volterra tube}} x^{-1/4} \|D_Z^2 f_Z\|_{\text{weighted}} < 20, \quad \sup x^{\beta+1} |g(x)|_c < 4 \cdot 10^{14}, \quad (33)$$

$$\sup_{0 \leq \theta \leq 1} x^2 \|D_Z f_Z[\alpha\zeta + \theta(U_4 - \alpha\zeta)] - D_Z f_Z[\alpha\zeta]\| < 70, \quad (34)$$

$$\sup x^2 \|\mathcal{R}\| < 100, \quad \inf \frac{U}{\zeta} > 1.0369. \quad (35)$$

It must additionally prove

$$\kappa x_0 > 26, \quad \Gamma_L < 0.08, \quad L_{\text{tail}} < 1, \quad B_{\text{tail}} < 1, \quad (36)$$

with the quantities in Section 6.

The direct segment bound (34) and the cone bound in (32) imply

$$\sup x^2 \|\mathcal{R}\| < 30 + 70 = 100. \quad (37)$$

The program encloses the separate family

$$U_\theta = \alpha\zeta + \theta(U_4 - \alpha\zeta), \quad 0 \leq \theta \leq 1, \quad (38)$$

on every parameter slab. If $B = w^{-9}Z[U_4 - \alpha\zeta]$, the scaled mean-value identity is

$$w^{-8}(Df[U_4] - Df[\alpha\zeta]) = \int_0^1 (wD^2f[U_\theta])[B] d\theta. \quad (39)$$

The code takes the product of the segment Hessian and base-coordinate norms on each individual box before maximising. The corresponding Arb upper endpoints are approximately

$$8.633982887330428 \quad \text{and} \quad 16.596375198438420,$$

respectively. Rounding outward to shorter safe decimals, the replay gives

$$\sup x^{-1/4} \|D_Z^2 f_Z[U_\theta]\|_{\text{weighted}} < 8.633983 < 20, \quad (40)$$

$$\sup x^2 \|D_Z f_Z[U_4] - D_Z f_Z[\alpha\zeta]\| < 16.596376 < 70. \quad (41)$$

It also proves $U_\theta/\zeta > 1.0369$ on the complete segment. Thus (34) is discharged independently of the Volterra-tube Hessian estimate.

6 Stable-base fixed point

The stable homogeneous solution normalised at x_0 is

$$\Phi_s(x) = e^{-\kappa(x-x_0)} \left(\frac{x}{x_0} \right)^{-5/4}. \quad (42)$$

For $|C_L| \leq \varepsilon_s = 10^{-11}$, decompose

$$Z = C_L \Phi_s e_s + Z_r, \quad (Z_r)_s(x_0) = 0. \quad (43)$$

In particular, $Z_s(x_0) = C_L$; C_L is a finite-base coordinate rather than an asymptotic Stokes coefficient.

For $H = (H_0, H_s, H_+, H_-)$, the diagonal Green operator is

$$(\mathcal{G}H)_0(x) = -x^{3/4} \int_x^\infty y^{-3/4} H_0(y) dy, \quad (44)$$

$$(\mathcal{G}H)_s(x) = \int_{x_0}^x e^{-\kappa(x-y)} \left(\frac{x}{y}\right)^{-5/4} H_s(y) dy, \quad (45)$$

$$(\mathcal{G}H)_\pm(x) = - \int_x^\infty e^{\lambda_\pm(x-y)} \left(\frac{x}{y}\right)^{-5/4} H_\pm(y) dy, \quad (46)$$

where $\lambda_\pm = \kappa e^{\pm i\pi/3}$. The centre and unstable components satisfy terminal conditions at infinity; the stable remainder has zero value at x_0 . With

$$\|Z_r\|_\beta = \sup_{x \geq x_0} x^\beta |Z_r(x)|_c, \quad \beta = \frac{57}{4}, \quad (47)$$

elementary integration gives

$$\|\mathcal{G}H\|_\beta \leq \Gamma_L \sup_{x \geq x_0} x^{\beta+1} |H(x)|_c, \quad \Gamma_L = \max \left\{ \frac{1}{\beta + 3/4}, \frac{2}{\kappa x_0} \right\}. \quad (48)$$

Let

$$\rho = \varepsilon_s + R x_0^{-\beta} = 10^{-11} + 10^{14} 20^{-19}. \quad (49)$$

Since $\kappa x_0 > 26 > \beta - 5/4$ and $u - 1 \geq \log u$ for $u = x/x_0 \geq 1$, the free stable orbit obeys

$$|\Phi_s(x)| \leq (x/x_0)^{-\beta}.$$

Consequently every point in the fixed-point ball satisfies

$$|C_L \Phi_s e_s + Z_r(x)|_c \leq \rho (x/x_0)^{-\beta},$$

which is the envelope used in the following Lipschitz and self-map bounds. Using the fresh chosen constants

$$K_{\text{lin}} = 100, \quad K_g = 4 \cdot 10^{14}, \quad K_N = 20, \quad R = 10^{14}, \quad (50)$$

the Lipschitz and self-map ratios are bounded by

$$L_{\text{tail}} \leq \Gamma_L \left(\frac{K_{\text{lin}}}{x_0} + 2K_N x_0^{5/4} \rho \right), \quad (51)$$

$$B_{\text{tail}} \leq \frac{\Gamma_L}{R} \left(K_g + K_{\text{lin}} \varepsilon_s x_0^{\beta-1} + K_N x_0^{\beta+5/4} \rho^2 \right) + \Gamma_L \frac{K_{\text{lin}}}{x_0}. \quad (52)$$

Theorem 6.1 (Fresh tail certificate). *The 256-bit $1024 \times 4 \times 4$ replay proves every strict inequality in Definition 5.1. The corresponding Arb upper endpoint for the complete linear constant is approximately 44.003106528084635. A safe shorter decimal upper bound is*

$$44.003107 < 100. \quad (53)$$

Consequently, for every (α, μ) in (28) and every $|C_L| \leq 10^{-11}$, the fixed-point equation

$$Z_r = \mathcal{G}[\mathcal{R}(C_L \Phi_s e_s + Z_r) + \mathcal{Q}(C_L \Phi_s e_s + Z_r) + g] \quad (54)$$

has a unique solution in $\|Z_r\|_\beta \leq 10^{14}$. This fixed point and its finite-base endpoint depend continuously (indeed, C^1) on (α, μ, C_L) throughout the validated parameter box. The corresponding profile solves (1) on $\zeta \geq 20$, satisfies

$$\frac{U(\zeta)}{\zeta} > 1.0369, \quad (55)$$

and its nonstable endpoint displacement is at most

$$\delta_\infty = R20^{-19} = 1.9073486328125 \cdot 10^{-11}. \quad (56)$$

Proof. The exact symbolic stage verifies (9)–(12) and the four forcing cancellations. The Arb stage evaluates all 16,384 rational slabs; the returned Boolean record includes the three successful segment checks

```
segment_weighted_hessian_lt_20,
segment_correction_lt_70,
segment_positive_ratio_gt_1_0369.
```

Equations (40)–(41) and the cone bound give (53).

The interval tests give the three hypotheses used in (51)–(52): the x^{-2} linear bound, the $x^{-\beta-1}$ forcing bound, and the quadratic Lipschitz bound on the complete tube. Equations (44)–(46) give (48). The strict inequality $L_{\text{tail}} < 1$ makes the Volterra map a contraction, and $B_{\text{tail}} < 1$ maps the closed radius- R ball strictly into itself. The Banach fixed-point theorem gives existence and uniqueness. These margins are uniform on the compact parameter box and the exact Volterra map is C^1 in its parameters. The parameter-dependent contraction theorem gives the asserted C^1 fixed point and endpoint. The interval positivity test applies to the entire tube, so it applies to the fixed point. Finally, $|Z_r(x_0)|_c \leq R x_0^{-\beta} = R20^{-19}$, which is (56). $\square$

Remark 6.2. The inherited endpoint number cannot be reused. The manuscript used $R = 5 \cdot 10^{13}$ and hence quoted $9.5367431640625 \cdot 10^{-12}$. The fresh proof chooses $R = 10^{14}$, so its displacement is twice as large. The signed-face calculation in Section 8 subtracts (56); using the inherited number would invalidate the face-sign enclosure.

7 Receipt and replay protocol

The master replay writes a canonical JSON object with schema `conical-pincher-certificate-v1` and status `proved`. It contains:

1. the embedded exact symbolic coefficients and cancellation record;
2. 256-bit precision and the $1024 \times 4 \times 4$ subdivision;
3. the exact parameter ranges and tube radius;
4. hashes of the generated coefficient, cone-remainder, and Hessian evaluators;
5. the shared-residual binding between the exact cancellation proof and the Arb forcing evaluator;

6. the Volterra-tube and cone-to- U_4 segment extrema;
7. every fresh computed extremum and derived scalar ratio;
8. every tail and centre Boolean acceptance test, all equal to `true`;
9. all 400 centre-step records at order 16, the parameter cube, positivity, the stable-coordinate bound, and the six signed faces;
10. a manifest of every proof-facing source with its aggregate digest.

Canonical JSON uses sorted keys, compact separators, ASCII encoding, and one terminal newline. Its SHA-256 digest is stored separately. The receipt, digest, exact source, dependency versions, recorded Python-build metadata, and source hashes form one indivisible record. After verification, `build_archive.py` writes a deterministic ZIP with fixed member metadata and an internal `MANIFEST.sha256`. That archive and its sibling digest form the reproducible certificate bundle for this calculation.

The 256-bit tail stage passes each grid test. In addition to (40), (41), and (53), its abbreviated displayed endpoints include

$$27.40673133, \quad 3.10193109423659 \cdot 10^{14}, \quad 8.63230045, \quad 1.92353999 \quad (57)$$

for the cone remainder, forcing, Volterra-tube scaled Hessian, and scaled base coordinate, respectively. Its positivity lower endpoint exceeded 1.0369, and its fixed-point ratios were approximately 0.1416 and 0.523. The receipt stores full outward-rounded records rather than these abbreviated decimal displays. On the author's machine this tail stage took about 50 seconds and used about 99 MB maximum resident memory; the separate 400-step centre validation took about 749 seconds.

The function `certify_tail()` itself calls the exact recursion checker, compiles $c_1, \dots, c_4$ from those exact expressions, and returns both the symbolic record and the forcing-cancellation binding.

The complete installation and replay commands are in `README.md`. They pin CPython 3.12, `python-flint==0.8.0`, and `sympy==1.13.3`. Exact rationals are passed directly to Arb; a Python binary floating-point number never enters an acceptance inequality.

The file `run_certificate.py` is the master script. It hashes the proof-facing source set before importing the local proof modules and again after the computation, runs the symbolic, tail, and centre stages, checks the centre–tail endpoint radius identity, and writes the canonical receipt and sibling digest. The standard-library program `verify_receipt.py` checks canonical encoding, the digest, the exact source manifest, the recorded dependency-version strings, every published threshold, the count and order of the 400 centre step records, and the six signed faces.

8 Validated centre certificate

The inherited nominal parameter vector is

$$p_0 = (0.63169185949612983, 0.030292801498271463, 1.037079401503446), \quad (58)$$

and the inherited affine cube is the parameter set

$$\mathcal{P} := p_0 + B[-r, r]^3, \quad r = 3 \cdot 10^{-11}, \quad (59)$$

where $B = (B_{ij})_{i,j=1}^3$ has the exact decimal-rational entries

$$\begin{aligned}
B_{11} &= -1.9874531069408703921803188384096016152 \cdot 10^{-8}, \\
B_{12} &= -3.89919252084496093216178406302010914955 \cdot 10^{-5}, \\
B_{13} &= 2.692876593501873590740705252497522410696 \cdot 10^{-4}, \\
B_{21} &= 9.93565749626417479846462113618818936 \cdot 10^{-11}, \\
B_{22} &= 2.812547048420073565711689937996722092401 \cdot 10^{-5}, \\
B_{23} &= -1.724902102072632493014507961965114922251 \cdot 10^{-5}, \\
B_{31} &= -5.0000228133190469535136000176975175576826 \cdot 10^{-2}, \\
B_{32} &= -6.146379043385490350935388358263219194 \cdot 10^{-4}, \\
B_{33} &= 1.2908993723726045408121737752290791486 \cdot 10^{-3}.
\end{aligned} \tag{60}$$

The proof-facing code uses the equivalent normalised form

$$\mathcal{P} = p_0 + rB[-1, 1]^3.$$

The three normalised cube variables are retained globally as degree-three Taylor-model symbols. The centre stage uses 400 equal time slabs of length $h = 1/20$, time Taylor order $N = 16$, and 256-bit Arb arithmetic. On one slab let

$$P(t, \xi) = \sum_{n=0}^N P_n(\xi) t^n, \quad 0 \leq t \leq h, \tag{61}$$

be the explicit polynomial generated from (5). Its degree-three parameter coefficients are Arb balls, and every discarded parameter monomial is accumulated in a symmetric Taylor-model remainder.

The time truncation is not treated as exact. To bound the omitted analytic tail uniformly at an arbitrary $\tau \in [0, h]$, the implementation performs the exact shift

$$[s^j]P(\tau + s, \xi) = \sum_{n=j}^N \binom{n}{j} P_n(\xi) \tau^{n-j}, \quad 0 \leq j \leq N. \tag{62}$$

In each shifted coefficient it retains the constant and three degree-one parameter coefficients and relaxes every degree-two and degree-three monomial, together with the existing Taylor-model error, into a symmetric affine remainder. Since $|\xi^\nu| \leq 1$ on $[-1, 1]^3$, this relaxation is an inclusion. The coefficient of s^{N+1} in the flux field composed with the shifted affine series, multiplied by $h^{N+1} = h^{17}$, is therefore an outward enclosure of the Taylor-theorem tail missing from the finite residual polynomial.

Write E_0 for the incoming componentwise error, D_i for the resulting complete residual bound in component i , and A_{ij} for bounds on the absolute flux Jacobian over a trial state tube. Each accepted slab exhibits positive radii T_i satisfying the strict componentwise inequalities

$$T_i > E_{0,i} + h \left(D_i + \sum_{j=1}^4 A_{ij} T_j \right), \quad i = 1, \dots, 4. \tag{63}$$

Lemma 8.1 (Correction-tube inclusion). *If the polynomial tube has $U > 0$ and (63) holds, then the exact solution of (5) entering with error bounded by E_0 remains in that tube throughout the slab.*

Proof. The vector field is locally Lipschitz wherever $U > 0$, hence the local solution is unique. If a component of the error first reached its boundary T_i , the integral form of the ODE and the mean-value theorem would give

$$|e_i(t)| \leq E_{0,i} + h \left(D_i + \sum_j A_{ij} T_j \right) < T_i,$$

a contradiction. Thus no first exit occurs. Note that this is a componentwise first-exit argument; the unweighted diagnostic quantity $h \|D_y f\|_\infty$ need not be smaller than one. $\square$

Applying Lemma 8.1 inductively over all 400 slabs proves that the degree-three endpoint models enclose the exact flux solution for the whole affine cube. The same correction tubes give

$$\inf_{0 \leq \zeta \leq 20, p \in \mathcal{P}} U(\zeta; p) > 0.971. \quad (64)$$

At $\zeta = 20$, the centre flux solution is converted to the physical four-jet, the four-term base jet is subtracted, and (19) is inverted. Let

$$F_c(p) = (Z_{0,c}, \Re Z_{+,c}, \Im Z_{+,c}), \quad C_L(p) = Z_{s,c}. \quad (65)$$

The exact matching function is

$$M(p) = F_c(p) - F_\infty(\alpha, \mu, C_L(p)), \quad (66)$$

where F_∞ is the nonstable endpoint of the tail fixed point. The centre certificate also proves

$$|C_L(p)| < 10^{-11}, \quad \max\{|Z_{0,c}(p_0)|, |\Re Z_{+,c}(p_0)|, |\Im Z_{+,c}(p_0)|\} < 5 \cdot 10^{-12}. \quad (67)$$

The second estimate includes the Taylor-model remainder and is a useful auxiliary centring check. The completed replay gives the sharper safe outward bounds

$$\inf U > 0.9996, \quad \sup |C_L| < 1.774939 \cdot 10^{-12}, \quad \|F_c(p_0)\|_\infty < 4.913763 \cdot 10^{-12}. \quad (68)$$

It also records a maximum complete slab defect below $3.929609 \cdot 10^{-25}$, a maximum shifted time tail below $2.497570 \cdot 10^{-25}$, and a maximum propagated solution error below $2.741703 \cdot 10^{-12}$. These are displayed roundings; the receipt retains the full Arb endpoints.

Theorem 6.1 supplies the uniform componentwise bound

$$|F_\infty|_c \leq 1.9073486328125 \cdot 10^{-11}; \quad (69)$$

the face calculation subtracts precisely this fresh radius, not the smaller historical radius.

For each receipt coordinate $i \in \{0, 1, 2\}$ and $\sigma \in \{-1, 1\}$, the code directly ranges the complete degree-three model for the i -th component of F_c on $\xi_i = \sigma$, with the other two variables in $[-1, 1]$. It includes the model remainder, subtracts the symmetric tail interval of radius $1.9073486328125 \cdot 10^{-11}$, and proves

$$\inf_{\xi_i = \sigma} \sigma M_i(p_0 + rB\xi) > 0. \quad (70)$$

These six direct signed ranges are the load-bearing numerical matching test. In coordinate/sign order their safe lower bounds are

receipt coordinate (zero-based)	sign	signed mismatch lower bound
0	−	$1.41413 \cdot 10^{-11}$
0	+	$7.18941 \cdot 10^{-12}$
1	−	$1.12970 \cdot 10^{-11}$
1	+	$1.03852 \cdot 10^{-11}$
2	−	$1.57006 \cdot 10^{-11}$
2	+	$6.01275 \cdot 10^{-12}$

Thus the minimum signed margin is greater than $6.01275 \cdot 10^{-12}$.

The validated global matching theorem is as follows.

Theorem 8.2. *There is a parameter $p = (\gamma, \mu, \alpha)$ in the cube (59) for which the positive centre solution and the positive tail solution match at $\zeta = 20$. Consequently they form a smooth positive solution of (1) on $[0, \infty)$ with positive asymptotic cone slope α .*

Proof. The validated centre integration gives (64) and encloses the endpoint models. Smooth parameter dependence for the centre flux ODE on the validated $U > 0$ tube makes its exact endpoint map continuous. The stable bound in (67) places every centre endpoint in the tail chart, so Theorem 6.1 supplies the corresponding continuous tail endpoint. The direct enclosures (70) give the six opposite face signs of the continuous matching map M . The Poincaré–Miranda theorem therefore supplies p with $M(p) = 0$. Equality of the four endpoint jet coordinates gives the smooth gluing, while (64) and Theorem 6.1 give positivity on the two halves. $\square$

The physical half-widths of the certified parameter cube are

$$\begin{aligned}
\Delta\gamma &\approx 9.248983772691192 \cdot 10^{-15}, \\
\Delta\mu &\approx 1.361237725845061 \cdot 10^{-15}, \\
\Delta\alpha &\approx 1.557172962297049 \cdot 10^{-12}.
\end{aligned} \tag{71}$$

The canonical receipt stores the exact rational values and the exact lower and upper hulls; the decimals in (71) are displays.

9 Record of constants

Quantity	Inherited manuscript	Present certificate
L	20	20 (reused)
tail truncation order N	4	4 (reused)
tail weight β	57/4	57/4 (reused)
parameter subdivision	$1024 \times 4 \times 4$	same (reused)
tail precision	256 bits	256 bits (reused)
K_{lin}	28	100 (fresh chosen)
K_g	$3.126 \cdot 10^{14}$	$4 \cdot 10^{14}$ (fresh chosen)
K_N	9	20 (fresh chosen)
R	$5 \cdot 10^{13}$	10^{14} (fresh chosen)
ε_s	10^{-11}	10^{-11} (reused)
tail positivity	1.0369879636	1.0369 (fresh threshold)
δ_∞	$9.5367431640625 \cdot 10^{-12}$	$1.9073486328125 \cdot 10^{-11}$
centre step/order	not printed	400/16 (fresh chosen)
centre parameter degree	not printed	3 (fresh chosen)
centre precision	192 bits	256 bits (fresh chosen)
centre residual threshold	not printed	$5 \cdot 10^{-12}$ (auxiliary)
matching faces	six	six direct signed ranges